Problem 1 (30 marks)
A certain Hamilton operator H is represented by the 3×3 matrix

$$\mathcal{H} = \hbar\omega \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ and the initial column } \psi_0 \equiv \psi(t=0) = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

represents the state ket $|\ \rangle$.

(a) Calculate the expectation value $\langle H \rangle$ and the spread δH of the Hamilton operator.
(b) Determine the eigenvalues, eigencolumns, and eigenrows of $\mathcal{H}$.
(c) If one measured H at $t = 0$, what would be the possible measurement results?
(d) With which probabilities would they occur?
(e) Calculate $\psi(t)$.

Problem 2 (15 marks)
Motion along the x axis. Kets $|a\rangle$ and $|b\rangle$ are specified by the Gaussian wave functions

$$\psi_a(x) = \langle x|a\rangle = \frac{(2\pi)^{-1/4}}{\sqrt{a}} \mathrm{e}^{-\left(\frac{x}{2a}\right)^2} \text{ with } a > 0\,,$$

$$\psi_b(x) = \langle x|b\rangle = \frac{(2\pi)^{-1/4}}{\sqrt{b}} \mathrm{e}^{-\left(\frac{x}{2b}\right)^2} \text{ with } b > 0\,.$$

(a) Calculate the transition probability $|\langle b|a\rangle|^2$.
(b) How large is it when $a = 2b$?

Problem 3 (15 marks)
Motion along the x axis; position operator X, momentum operator P.
(a) Evaluate the commutator $[X, \mathrm{e}^{\mathrm{i}xP/\hbar}]$ between the position operator X and the unitary displacement operator $\mathrm{e}^{\mathrm{i}xP/\hbar}$ (where x is any real *number*).
(b) Use this commutator (or any other method) to show that

$$\mathrm{e}^{-\mathrm{i}xP/\hbar} X \mathrm{e}^{\mathrm{i}xP/\hbar} = X - x\,.$$

Problem 4 (20 marks)
Motion along the x axis; position operator X, momentum operator P. For small displacements x, the squared expectation value of the unitary displacement operator $\mathrm{e}^{\mathrm{i}xP/\hbar}$ is of the form $\left|\left\langle \mathrm{e}^{\mathrm{i}xP/\hbar} \right\rangle\right|^2 = 1 - (x/L)^2 + O(x^4)$, where L is the so-called *coherence length*.
(a) How is L related to the momentum spread δP?
(b) Which upper bound on L is set by the position spread δX?

Problem 5 (20 marks)
In terms of the ladder operators $A^\dagger$ and A, the Hamilton operator of a harmonic oscillator is $H = \hbar\omega A^\dagger A$.
(a) State the Heisenberg equations of motion for $A(t)$ and $A^\dagger(t)$.
(b) Solve them, that is: express $A(t)$ and $A^\dagger(t)$ in terms of $A(0)$ and $A^\dagger(0)$.
(c) Then evaluate the commutator $[A(t_1), A^\dagger(t_2)]$, where t_1 and t_2 are two arbitrary instants.