Problem 1 (20 points)
Motion along the x axis. State 1 is specified by its position wave function

$$\langle x|1\rangle = \frac{(2\pi)^{-1/4}}{\sqrt{a}} \mathrm{e}^{-\left(\frac{x}{2a}\right)^2} \qquad \text{with } a > 0,$$

and state 2 is specified by its momentum wave function

$$\langle p|2\rangle = \frac{(2\pi)^{-1/4}}{\sqrt{b}} \mathrm{e}^{-\left(\frac{p}{2b}\right)^2} \qquad \text{with } b > 0.$$

Calculate the transition probability $|\langle 1|2\rangle|^2$. [Hint: You can run a simple check on your answer, because you know the probability when $2ab = \hbar$.]

Problem 2 (25 points)
Motion along the x axis; position operator X, momentum operator P.
Consider the Hamilton operator $H = -\Omega(XP + PX)$ with $\Omega > 0$.

(a) Solve the Heisenberg equations of motion, that is: express $X(t)$ and $P(t)$ in terms of $X(t_0)$, $P(t_0)$, and $T = t - t_0$.

(b) Evaluate the commutator $[X(t), P(t_0)]$.

(c) Find first the time transformation function $\langle x, t|p, t_0\rangle$ and then the time transformation function $\langle x, t|x', t_0\rangle$.

Problem 3 (15 points)
Harmonic oscillator; ladder operators $A^\dagger$ and A; Hamilton operator $H = \hbar\omega A^\dagger A$.
At the initial time t_0, the statistical operator $\rho(A^\dagger, A, t_0)$ is given by the normally ordered exponential

$$\rho(A^\dagger, A, t_0) = \mathrm{e}^{-(A^\dagger - \alpha^*);(A - \alpha)},$$

where α is an arbitrary complex number and α^* is its complex conjugate. What is the statistical operator $\rho(A^\dagger, A, t)$ at the later time $t = t_0 + T$?

Problem 4 (20 points)
A harmonic oscillator is in the state described by the ket $|\ \rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, which is an equal-weight superposition of the Fock states to $n = 0$ and $n = 1$.

(a) What are the expectation values of A, $A^\dagger$, A^2, and ${A^\dagger}^2$?

(b) What are the spreads δX, δP of position operator X and momentum operator P? How large is their product $\delta X\,\delta P$?

Problem 5 (20 points)

(a) Show that

$$\mathrm{tr}\{F\} = \int \frac{\mathrm{d}x\mathrm{d}p}{2\pi\hbar}\, f\Big(-\mathrm{i}\frac{\ell p}{\hbar}, \frac{x}{\ell}\Big)$$

where $F = f(A^\dagger, A)$ is the normally ordered form of operator F.

(b) Use this to calculate the trace of $\mathrm{e}^{-\lambda A^\dagger;A}$ with $\lambda > 0$.