1. The statistical operator of a one-dimensional harmonic oscillator (with ladder operators A, $A^\dagger$ and Fock state kets $|n\rangle$) is

$$\rho = Z(\beta)\,\mathrm{e}^{-\beta A^\dagger A} = Z(\beta)\sum_{n=0}^{\infty} |n\rangle\,\mathrm{e}^{-n\beta}\,\langle n| \qquad \text{with } \beta > 0.$$

(a) Determine the normalization factor $Z(\beta)$. (8 marks)

(b) Use a basic property of the trace, and a basic property of the ladder operators, to show that the expectation values of powers of $A^\dagger A$ obey

$$\left\langle \left(A^\dagger A\right)^k \right\rangle = \mathrm{e}^{-\beta} \left\langle \left(A^\dagger A + 1\right)^k \right\rangle$$

for $k = 1, 2, 3, \ldots$. (10 marks)

(c) Find the expectation value and the spread of $A^\dagger A$, either by exploiting the identity of part (b) or by any other method. (7 marks)

2. Orbital angular momentum: vector operator $\vec{L}$ with components L_1, L_2, and L_3. As usual, we denote by $|l, m\rangle$ the joint eigenkets of $\vec{L}^2$ and L_3 (eigenvalue $\hbar^2 l(l+1)$ of $\vec{L}^2$; eigenvalue $\hbar m$ of L_3).

(a) Determine the spreads of L_1 and L_2 in the state with ket $|l, m\rangle$. For given l, what are the largest and smallest values of these spreads? (10 marks)

(b) Consider the $l = 1$ superposition state

$$|\;\rangle = \frac{1}{\sqrt{2}}\Big(|1, 1\rangle + |1, -1\rangle\Big)\,.$$

Determine the expectation values of L_1, L_2, L_3 as well as L_1^2, L_2^2, L_3^2, and then their spreads δL_1, δL_2, δL_3. (15 marks)

[Hint: Remember about the ladder operators $L_\pm = L_1 \pm \mathrm{i}L_2$.]

3. Consider the one-dimensional Hamilton operator

$$H = \frac{1}{2M}P^2 - \frac{(\hbar\kappa)^2}{\sqrt{2}\,M}\,\mathrm{e}^{-\frac{1}{2}(\kappa X)^2}\,,$$

where X is the particle's position operator, P is its momentum operator, M is its mass, and $\kappa > 0$ determines the strength and range of the Gaussian potential energy.

(a) Determine the expectation values of the kinetic and of the potential energy,

$$E_{\mathrm{kin}} = \frac{1}{2M}\left\langle P^2 \right\rangle \quad \text{and} \qquad E_{\mathrm{pot}} = -\frac{(\hbar\kappa)^2}{\sqrt{2}\,M}\left\langle \mathrm{e}^{-\frac{1}{2}(\kappa X)^2} \right\rangle\,,$$

for a Gaussian wave function $\psi(x) = \langle x|\ \rangle = \dfrac{(2\pi)^{-1/4}}{\sqrt{a}}\,\mathrm{e}^{-\left(\frac{x}{2a}\right)^2}$ (with $a > 0$). Write E_{kin} and E_{pot} as multiples of $(\hbar\kappa)^2/(2M)$. (15 marks)

(b) Use them to get an upper bound on the ground-state energy E_0 of the Hamilton operator H. [Hint: Remember the Rayleigh–Ritz variational principle; optimize the value of $y \equiv (\kappa a)^2$.] (10 marks)

4. The Hamilton operator

$$H = \hbar\omega A^\dagger A + \frac{1}{2}\mathrm{i}\hbar\Omega({A^\dagger}^2 - A^2) \qquad \text{with} \quad -\omega < \Omega < \omega$$

is that of a one-dimensional harmonic oscillator (ladder operators $A^\dagger$ and A, circular frequency ω), with a perturbation proportional to ${A^\dagger}^2 - A^2$ of strength Ω.

(a) For $n = 0, 1, 2, \ldots$, find the n-th eigenvalue of H to second order in Ω by an application of Rayleigh–Schrödinger perturbation theory. (12 marks)

(b) Determine the second-order approximation to the ground-state energy in Brillouin–Wigner perturbation theory. (8 marks)

(c) Compare your results of parts (a) and (b) with the exact energy eigenvalues $E_n = (n + \frac{1}{2})\hbar\sqrt{\omega^2 - \Omega^2} - \frac{1}{2}\hbar\omega$. (5 marks)