Problem 1 (25=10+10+5 points)
The unitary operators U and V are the standard complementary pair of cyclic permutators of period N for an N-dimensional quantum degree of freedom. We consider odd N values only, that is: $N = 2M + 1$ with $M = 1, 2, 3, \ldots$, and define N^2 operators in accordance with

$$W_{00} = \frac{1}{N} \sum_{j=-M}^{M} \sum_{k=-M}^{M} U^j V^k \mathrm{e}^{\mathrm{i}\pi jk/N} \qquad \text{and} \quad W_{lm} = V^m U^{-l} W_{00} U^l V^{-m} .$$

for $-M \le l, m \le M$.

(a) Show that W_{00} is hermitian. Then conclude that all W_{lm}s are hermitian.

(b) Evaluate $\mathrm{tr}\left\{W_{lm}\right\}$ and $\mathrm{tr}\left\{W_{lm} W_{l'm'}\right\}$.

(c) Establish that an arbitrary operator F can be written as a weighted sum of the W_{lm}s,

$$F = \frac{1}{N} \sum_{l,m} f_{lm} W_{lm} \qquad \text{with} \quad f_{lm} = \mathrm{tr}\left\{F W_{lm}\right\} ,$$

and express $\mathrm{tr}\left\{F\right\}$ in terms of the coefficients f_{lm}.

Problem 2 (25=10+15 points)
Motion along the x axis; position operator X, momentum operator P. The system is described by the statistical operator $\rho = \rho(X, P)$. In

$$\begin{aligned} R &\overset{[1]}{=} \int \frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar} \rho(X + x, P + p) \\ &\overset{[2]}{=} \int \frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar}\, \mathrm{e}^{-\mathrm{i}pX/\hbar} \mathrm{e}^{\mathrm{i}xP/\hbar} \rho(X, P)\, \mathrm{e}^{-\mathrm{i}xP/\hbar} \mathrm{e}^{\mathrm{i}pX/\hbar} \overset{[3]}{=} 1 , \end{aligned}$$

equal sign [1] defines operator R, equal sign [2] is an identity, and equal sign [3] needs to be demonstrated. Therefore,

(a) show that [2] holds; and

(b) demonstrate [3]. (Hint: Consider $\langle x'|R|p'\rangle$.)

Problem 3 (25=15+10 points)
Motion along the x axis; position operator X, momentum operator P. The dynamics is governed by the Hamilton operator

$$H = \frac{1}{2M}P^2 + \frac{1}{2}\gamma(XP + PX)$$

with mass M and rate constant γ.

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}t}(XP + PX) = \frac{2}{M}P^2$ and use this to find $X(t)P(t) + P(t)X(t)$ in terms of $X(t_1)$ and $P(t_2)$.

(b) Then employ the quantum action principle to determine first $\delta_\gamma\langle x, t_1|p, t_2\rangle$ and then $\langle x, t_1|p, t_2\rangle$.

Problem 4 (25=8+7+10 points)
Function $f(a)$ is related to its Fourier transform $g(\alpha)$ by $f(a) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi}\, g(\alpha)\, \mathrm{e}^{\mathrm{i}\alpha a}$.
Operator A is such that $f(A)$ is well defined.

(a) Show that

$$f'(A) = \frac{\mathrm{d}f(A)}{\mathrm{d}A} = \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi}\, \mathrm{i}\alpha g(\alpha)\, \mathrm{e}^{\mathrm{i}\alpha A}$$

and

$$\delta f(A) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi}\, g(\alpha) \int_0^{\alpha} \mathrm{d}\beta\, \mathrm{e}^{\mathrm{i}\beta A}\, \mathrm{i}\delta A\, \mathrm{e}^{\mathrm{i}(\alpha-\beta)A}\,.$$

(b) When is $\delta f(A) = f'(A)\,\delta A$ true?

(c) Show that $\mathrm{tr}\,\{\delta f(A)\} = \mathrm{tr}\,\{f'(A)\,\delta A\}$ holds even if $\delta f(A) \neq f'(A)\,\delta A$.