Problem 1 (20=6+8+6 points)
A harmonic oscillator (natural frequency ω, ladder operators A, $A^\dagger$, unperturbed Hamilton operator $H_0 = \hbar\omega A^\dagger A$) is exposed to a time-independent perturbation that is specified by $H_1 = \hbar(\Omega A^\dagger + \Omega^* A)$. At the initial time $t = 0$, the oscillator is in the ground state of H_0.

(a) How large is the energy spread δH for $H = H_0 + H_1$?

(b) What is the scattering operator $S(T) = \mathrm{e}^{\mathrm{i}H_0 T/\hbar}\mathrm{e}^{-\mathrm{i}HT/\hbar}$ to first order in Ω?

(c) What is the probability, to lowest order in Ω, for finding the oscillator in the 1st, 2nd, 3rd, ... excited state of H_0 after time T has elapsed?

Problem 2 (25=5+8+5+7 points)
The state of a particle of mass M is described by the wave function

$$\psi(\vec{r}, t) = C\,\frac{x + \mathrm{i}y}{r}\,\mathrm{e}^{-r/a}\,\mathrm{e}^{-\mathrm{i}\omega t}\,,$$

where $C > 0$, $a > 0$, and $\omega > 0$.

(a) Determine the normalization constant C.

(b) Find the probability density $\rho(\vec{r}, t)$
and the probability current density $\vec{j}(\vec{r}, t)$.

(c) Verify that they obey the continuity equation.

(d) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int(\mathrm{d}\vec{r})\,\vec{r}\rho(\vec{r}, t) = \int(\mathrm{d}\vec{r})\,\vec{j}(\vec{r}, t)$$

holds, either by a general argument or by an explicit calculation for the $\rho(\vec{r}, t)$ and $\vec{j}(\vec{r}, t)$ of part (b).
Hint: Remember that $\vec{s}\cdot\vec{\nabla}\,\vec{r} = \vec{s}$ for any 3-vector $\vec{s}$.

Problem 3 (25=12+8+5 points)
A particle of mass M is scattered by the Gaussian potential

$$V(\vec{r}) = V_0\,\mathrm{e}^{-\frac{1}{2}(r/a)^2} \quad \text{with } a>0 \quad \text{and } V_0 = \frac{(\hbar/a)^2}{2M}\,.$$

Apply the first-order Born approximation and determine

(a) the differential cross section $\dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta)$ in terms of a and $q = 2k\sin\dfrac{\theta}{2}$;

(b) the total cross section σ in terms of a and E/V_0 with $E = \dfrac{(\hbar k)^2}{2M}$;

(c) the dominating E dependence for $E \ll V_0$ and $E \gg V_0$.

Hint: Remember that $\mathrm{d}\theta\,\sin\theta = \dfrac{1}{k^2}\mathrm{d}q\,q$.

Problem 4 (30=5+10+10+5 points)
Two-level atom; probability amplitudes for states 1 and 2 are ψ_1 and ψ_2, respectively.
The Schrödinger equation for $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ is $\mathrm{i}\hbar\dfrac{\partial}{\partial t}\psi(t) = \mathcal{H}(t)\psi(t)$ with

$$\mathcal{H}(t) = \hbar\omega\begin{pmatrix}\cos(2\phi(t)) & \sin(2\phi(t))\\ \sin(2\phi(t)) & -\cos(2\phi(t))\end{pmatrix} \quad \text{where } \phi(t) = \pi t/T$$

with $T>0$.

(a) Find the eigencolumns $\psi_\pm(t)$ of $\mathcal{H}(t)$ to the eigenvalues $\pm\hbar\omega$.

(b) Write $\psi(t) = \alpha(t)\psi_+(t) + \beta(t)\psi_-(t)$ and find the 2×2 matrix $\mathcal{M}$ in

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = \mathrm{i}\,\mathcal{M}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}\,.$$

[Check: If you get it right, $\mathcal{M}$ does not depend on t.]

(c) By solving this equation, find $\psi(T)$ for $\psi(0) = \begin{pmatrix}1\\0\end{pmatrix}$.

(d) What is the dominating T dependence of the probability $|\psi_2(T)|^2$ for $\omega T \ll 1$? And what is it for $\omega T \gg 1$?