1. A particle of mass M and kinetic energy $E = \dfrac{(\hbar k)^2}{2M}$ is scattered by the δ-shell potential

$$V(\vec{r}) = V_0\, a\, \delta(r-a) \quad \text{with } a > 0 \quad \text{and } V_0 = \frac{(\hbar/a)^2}{2M}\,.$$

Determine the total cross section σ_0 for s-wave scattering by the following steps.

(a) Justify the general form $u_0(r) = \begin{cases} C\sin(kr) & \text{for } 0 < r < a \\ \sin(kr+\delta_0) & \text{for } r > a \end{cases}$

of the radial wave function, and show that the phase shift δ_0 and the amplitude factor C are determined by

$$\sin(ka+\delta_0) = C\sin(ka)\,, \qquad \cos(ka+\delta_0) = C\left[\cos(ka) + \frac{\sin(ka)}{ka}\right]\,.$$

[10 marks]

(b) Use these equations to find the function $f(ka)$ in $\sigma_0 = \pi a^2 f(ka)$.

[10 marks]

(c) What is the dominating term when $ka \ll 1$? [5 marks]

2. The angular momentum vector operator of a spin-$\frac{1}{2}$ particle is $\vec{J} = \vec{L} + \vec{S}$, where $\vec{L} = \vec{R} \times \vec{P}$ and $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ are the vector operators for orbital angular momentum and spin, respectively. As usual, we denote the eigenvalues of $\vec{L}^2$ and $\vec{J}^2$ by $l(l+1)\hbar^2$ and $j(j+1)\hbar^2$, and those of J_z, L_z, S_z by $m\hbar$, $m_l\hbar$, $m_s\hbar$.

In the following, the value of l is arbitrary, but fixed.

(a) Show that the joint eigenkets $|j,m\rangle$ of $\vec{J}^2$ and J_z are also eigenkets of $\vec{L}\cdot\vec{S}$ with eigenvalues $\frac{1}{2}l\hbar^2$ for $j = l+\frac{1}{2}$ and $-\frac{1}{2}(l+1)\hbar^2$ for $j = l-\frac{1}{2}$. [9 marks]

(b) Express the projectors on the subspaces to $j = l \pm \frac{1}{2}$ as simple functions of $\vec{L}\cdot\vec{S}$.

[8 marks]

(c) When the system is prepared in a common eigenstate of L_z and S_z, what is the probability for finding $j = l + \frac{1}{2}$? [8 marks]

[Check: For $(m_l, m_s) = (l, \frac{1}{2})$ or $(m_l, m_s) = (-l, -\frac{1}{2})$, the probability in (c) is 1.]

3. A harmonic oscillator (natural frequency ω, ladder operators A, $A^\dagger$, unperturbed Hamilton operator $H_0 = \hbar\omega A^\dagger A$) is exposed to a time-independent perturbation that is specified by

$$H_1 = \hbar\Omega(A^\dagger + A)^3 \qquad \text{with} \quad \Omega > 0\,.$$

At the initial time $t = 0$, the oscillator is in the ground state of H_0.

(a) For short times, the probability $\mathrm{prob}(0 \to 0, t)$ for remaining in the ground state of H_0 is of the form $\mathrm{prob}(0 \to 0, t) = 1 - (\gamma t)^2$. Determine the value of γ. [10 marks]

(b) Write $\overline{H_1}(t) = \mathrm{e}^{\mathrm{i}H_0 t/\hbar} H_1 \mathrm{e}^{-\mathrm{i}H_0 t/\hbar}$ as an explicit polynomial in $A^\dagger$ and A. [6 marks]

(c) What is the probability, to lowest order in Ω, for finding the oscillator in the 1st, 2nd, 3rd, ... excited state of H_0 after time T has elapsed? [9 marks]

4. Consider a two-level atom with the probability amplitudes for states 1 and 2 denoted by ψ_1 and ψ_2, respectively. The Schrödinger equation for $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ is $\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(t) = \mathcal{H}(t)\psi(t)$ where the 2×2 Hamilton matrix $\mathcal{H}(t)$ has the time-independent eigenvalues $\pm\hbar\omega$ and the time-dependent eigencolumns

$$\psi_\pm(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{e}^{2\pi\mathrm{i}t/T}\\ \pm\mathrm{e}^{-2\pi\mathrm{i}t/T}\end{pmatrix} \qquad \text{with} \quad T > 0\,.$$

(a) Write $\psi(t) = \alpha(t)\psi_+(t) + \beta(t)\psi_-(t)$ and find the 2×2 matrix $\mathcal{M}$ in

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = \mathrm{i}\,\mathcal{M}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}\,.$$

[Check: If you get it right, $\mathcal{M}$ does not depend on t.] [8 marks]

(b) By solving this equation, find $\alpha(t)$ and $\beta(t)$ for $\alpha(0) = 1$, $\beta(0) = 0$. [10 marks]

(c) What is the dominating T dependence of the probability $|\beta(t = T)|^2$ for $\omega T \ll 1$? And what is it for $\omega T \gg 1$? [7 marks]