Sample Solution to Problem 1
For $u(x,y) = \cos x\,\cosh y$, the Cauchy–Riemann relations state

$$\begin{aligned}\frac{\partial}{\partial x}v(x,y) &= -\frac{\partial}{\partial y}u(x,y) = -\cos x\,\sinh y\,,\\ \frac{\partial}{\partial y}v(x,y) &= \frac{\partial}{\partial x}u(x,y) = -\sin x\,\cosh y\,.\end{aligned}$$

Taken together, these imply $v(x,y) = -\sin x\,\sinh y + c$ where c is real and independent of x and y, but otherwise arbitrary. Therefore

$$\begin{aligned}f(z) &= u(x,y) + \mathrm{i}v(x,y)\\ &= \cos x\,\cosh y - \mathrm{i}\sin x\,\sinh y + \mathrm{i}c = \cos(x+\mathrm{i}y) + \mathrm{i}c\\ &= \mathrm{i}c + \cos z\,.\end{aligned}$$

Sample Solution to Problem 2
Since $f(z)$ is real when $z = x + \mathrm{i}0 = x$ is real, all its derivatives are real at $z = 0$,

$$a_n = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n f(z)\Big|_{z=0} = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x)\Big|_{x=0} = {a_n}^*\,.$$

Accordingly, the Taylor expansion at $z = 0$ gives

$$f(z)^* = \left(\sum_{n=0}^{\infty} a_n z^n\right)^* = \sum_{n=0}^{\infty} {a_n}^*(z^*)^n = \sum_{n=0}^{\infty} a_n(z^*)^n = f(z^*)\,,$$

indeed.

Sample Solution to Problem 3
For (1) and (3) we use $z = x$, $\mathrm{d}z = \mathrm{d}x$ for the parameterization:

$$\int_{z_1}^{z_2}\mathrm{d}z\,|z|^2 = \int_{-1}^{-r}\mathrm{d}x\,x^2 = \frac{1}{3}(1-r^3)\,,\qquad \int_{z_3}^{z_4}\mathrm{d}z\,|z|^2 = \int_{r}^{1}\mathrm{d}x\,x^2 = \frac{1}{3}(1-r^3)\,.$$

For (2) we use $z = r\mathrm{e}^{\pm\mathrm{i}\varphi}$, $\mathrm{d}z = \mathrm{d}\varphi\,(\pm\mathrm{i}r)\mathrm{e}^{\pm\mathrm{i}\varphi}$ for the parameterization:

$$\int_{z_2}^{z_3}\mathrm{d}z\,|z|^2 = \int_{-\pi}^{0}\mathrm{d}\varphi\,(\pm\mathrm{i}r)\mathrm{e}^{\pm\mathrm{i}\varphi}\,r^2 = 2r^3\,.$$

Together then

$$\int_{z_1}^{z_4}\mathrm{d}z\,|z|^2 = \frac{2}{3}(1-r^3) + 2r^3 = \frac{2}{3} + \frac{4}{3}r^3\,.$$

Sample Solution to Problem 4
The singularities are a 2nd-order pole at $z_1 = 2$ and a 1st-order pole at $z_2 = -2$. The respective residues are

$$r_1 = \frac{\mathrm{d}}{\mathrm{d}z}(z - z_1)^2 f(z)\Big|_{z=z_1} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{e}^{-\mathrm{i}z}}{z+2}\Big|_{z=2} = -\frac{1+4\mathrm{i}}{16}\,\mathrm{e}^{-2\mathrm{i}}\,,$$

$$r_2 = (z - z_2) f(z)\Big|_{z=z_2} = \frac{\mathrm{e}^{-\mathrm{i}z}}{(z-2)^2}\Big|_{z=-2} = \frac{1}{16}\,\mathrm{e}^{2\mathrm{i}}\,.$$

Sample Solution to Problem 5
We put $\sin\varphi = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{-\mathrm{i}\varphi}\right) = \frac{1}{2\mathrm{i}}(z - 1/z) = \frac{1}{2\mathrm{i}z}(z^2 - 1)$ with $z = \mathrm{e}^{\mathrm{i}\varphi}$ and $\mathrm{d}\varphi = \frac{\mathrm{d}z}{\mathrm{i}z}$. This converts the φ integral into a z integral over the unit circle,

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\,\frac{\cosh t}{\sinh t + \mathrm{i}\sin\varphi} &= \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}z}\,\frac{\cosh t}{\sinh t + \frac{1}{2z}(z^2-1)} \\ &= \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\,\underbrace{\frac{2\cosh t}{z^2 + 2z\sinh t - 1}}_{=f(z)} \\ &= (\text{sum of the residues of } f(z) \text{ to its poles} \\ &\qquad \textit{inside} \text{ the unit circle})\,.\end{aligned}$$

Since $2\sinh t = \mathrm{e}^t - \mathrm{e}^{-t}$ the denominator of $f(z)$ can be written as

$$z^2 + 2z\sinh t - 1 = (z + \mathrm{e}^t)(z - \mathrm{e}^{-t})\,,$$

so that $f(z)$ has 1st-order poles at $z_1 = -\mathrm{e}^t$ and $z_2 = \mathrm{e}^{-t}$ with the residues

$$r_1 = (z - z_1) f(z)\Big|_{z=z_1} = \frac{2\cosh t}{-\mathrm{e}^t - \mathrm{e}^{-t}} = -1\,,$$

$$r_2 = (z - z_2) f(z)\Big|_{z=z_2} = \frac{2\cosh t}{\mathrm{e}^{-t} + \mathrm{e}^t} = 1\,.$$

Only z_2 is inside the unit circle for $t > 0$, and for $t < 0$ it's only z_1. Therefore we obtain

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\,\frac{\cosh t}{\sinh t + \mathrm{i}\sin\varphi} = \left\{\begin{array}{rl} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{array}\right\} = \operatorname{sgn} t\,.$$