PART I

Answer ALL of the following FIVE short questions.

Short Question S1 (8 marks)
Your friend told you that $v(x, y) = x^2 + 2xy - y^2$ is the imaginary part of an entire function $f(z)$ of $z = x + \mathrm{i}y$. You also know that $f(0) = 0$. Determine the real part $u(x, y)$ of $f(z)$. Then write $f(z)$ compactly.

Short Question S2 (8 marks)
Evaluate the integral

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \delta(x^2 - 4x + 3) \, \mathrm{e}^{\mathrm{i}\pi x} \, .$$

Short Question S3 (8 marks)
The scalar field $\Phi(\mathbf{r})$ is given by $\Phi(\mathbf{r}) = (\mathbf{b} \times \mathbf{r})^2$, whereby $\mathbf{b}$ is a constant vector, that is: $\mathbf{b}$ does not depend on the position vector $\mathbf{r}$. Find $\nabla\Phi$ and $\nabla^2\Phi$.

Short Question S4 (8 marks)
Volume V is bounded by surface S; $\mathbf{F}(\mathbf{r})$ is a vector field. Show that

$$\int_S \mathrm{d}\mathbf{S} \; \nabla \cdot \mathbf{F} = \int_V (\mathrm{d}\mathbf{r}) \; [\nabla \times (\nabla \times \mathbf{F}) + \nabla^2 \, \mathbf{F}] \, .$$

Short Question S5 (8 marks)
The function $y(x)$ obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} \, .$$

If $y(0) = 3$, what is $y(4)$?

PART II

Answer ANY TWO OF the following three long questions.

Long Question L1 (30 marks)
Calculate the Fourier transform $f(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} F(t)$ of $F(t) = \dfrac{1}{(t^2+T^2)^2}$ with $T > 0$ by a suitable contour integration. Verify that $f(\omega = 0)$ has the correct value by calculating $\int_{-\infty}^{\infty} \mathrm{d}t\, F(t)$ as the T derivative of a known integral.

Long Question L2 (30 marks)
Three given vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are such that $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} > 0$.
(a) Show that $[(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})]\cdot(\mathbf{c}\times\mathbf{a}) = [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]^2$.
(b) Three more vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are defined by

$$\begin{array}{lll} \mathbf{a}\cdot\mathbf{a}' = 1\,, & \mathbf{a}\cdot\mathbf{b}' = 0\,, & \mathbf{a}\cdot\mathbf{c}' = 0\,, \\ \mathbf{b}\cdot\mathbf{a}' = 0\,, & \mathbf{b}\cdot\mathbf{b}' = 1\,, & \mathbf{b}\cdot\mathbf{c}' = 0\,, \\ \mathbf{c}\cdot\mathbf{a}' = 0\,, & \mathbf{c}\cdot\mathbf{b}' = 0\,, & \mathbf{c}\cdot\mathbf{c}' = 1\,. \end{array}$$

Express $\mathbf{a}'$, $\mathbf{b}'$, and $\mathbf{c}'$ in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.
(c) What are the cartesian coordinates of $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ if those of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are given by

$$\mathbf{a} \mathrel{\widehat{=}} \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \mathbf{b} \mathrel{\widehat{=}} \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \mathbf{c} \mathrel{\widehat{=}} \begin{pmatrix}1\\1\\0\end{pmatrix}?$$

Long Question L3 (30 marks)
The generating function $G(t,\varphi)$ for the Bessel functions $J_m(t)$ is

$$G(t,\varphi) = \mathrm{e}^{\mathrm{i}t\cos\varphi} = \sum_{m=-\infty}^{\infty} \mathrm{i}^m \mathrm{e}^{\mathrm{i}m\varphi} J_m(t)\,.$$

(a) Which three symmetry properties of the $J_m(t)$ follow from the three identities

$$G(t,\varphi) = G(t,\pi-\varphi)^*\,, \quad G(t,\varphi) = G(t,-\varphi)\,, \quad G(t,\varphi) = G(-t,\pi+\varphi)\,?$$

(b) Derive the recurrence relations

$$\frac{2m}{t}J_m(t) = J_{m-1}(t) + J_{m+1}(t)\,, \qquad 2\frac{\mathrm{d}}{\mathrm{d}t}J_m(t) = J_{m-1}(t) - J_{m+1}(t)\,.$$

[Hint: Differentiate $G(t,\varphi)$.]