Problem 1 (25 marks)
The state of a system is specified by its momentum wave function

$$\psi(p) = \langle p|\ \rangle = \begin{cases} C\,(p_0 - |p|) & \text{for } 0 \le |p| \le p_0\,, \\ 0 & \text{for } 0 < p_0 \le |p|\,. \end{cases}$$

First, determine the normalization constant $C > 0$ in terms of p_0. Then, express the expectation values $\langle X\rangle$, $\langle X^2\rangle$, $\langle P\rangle$, $\langle P^2\rangle$ as integrals involving $\psi(p)$. Now, calculate the spreads δX and δP and verify that they obey the uncertainty relation.

Problem 2 (25 marks)
Operator F is a function of position operator X and momentum operator P that is specified by

$$\langle x|F|p\rangle = \mathrm{e}^{-\frac{1}{2}(x/x_0)^2 - \frac{1}{2}(p/p_0)^2}\,,$$

where x_0 and p_0 are positive constants. Calculate the traces $\mathrm{tr}\{F\}$, $\mathrm{tr}\{F^\dagger\}$, and $\mathrm{tr}\{F^\dagger F\}$.

Problem 3 (25 marks)
Operator Γ is a function of position operator X and momentum operator P that is given by

$$\Gamma = aP/\hbar + (X/a)^2 \qquad \text{where } a > 0 \text{ is a constant length parameter}\,.$$

The eigenvalues γ of Γ comprise all real numbers, $\Gamma|\gamma\rangle = |\gamma\rangle\gamma$. Find the position wave function $\langle x|\gamma\rangle$ to eigenket $|\gamma\rangle$ with the convention

$$\langle x|\gamma\rangle = (\text{a } a \text{ dependent pre-factor}) \times (\text{a function of } \gamma \text{ and } x/a),$$

with the pre-factor determined by the normalization condition $\langle\gamma|\gamma'\rangle = \delta(\gamma - \gamma')$.

Problem 4 (25 marks)
A particle (mass M, position operator X, momentum operator P) moves along the x axis under the influence of the Hamilton operator $H = \dfrac{1}{2M}(P + M\omega X)^2$ where $\omega > 0$ is a constant frequency parameter.

(a) State the Heisenberg equations of motion for $P(t)$ and $X(t)$, and solve them.

(b) Use these solutions to evaluate the commutators

$$[X(t_0), X(t)]\,, \quad [P(t_0), P(t)]\,, \quad [X(t_0), P(t)]\,, \quad [P(t_0), X(t)]$$

in terms of the elapsed time $T = t - t_0$.