These sample solutions were prepared by Bess Fang.

Problem 1

A harmonic oscillator is in the coherent state described by the ket $|a\rangle$. In order to compute the expectation value of position X and momentum P and their spreads δX and δP, we first re-write these operators in terms of the ladder operators A and $A^\dagger$:

$$
\begin{aligned}
X &= \frac{l}{\sqrt{2}}(A^\dagger + A)\,, \\
P &= \frac{\hbar/l}{\sqrt{2}}(\mathrm{i}A^\dagger - \mathrm{i}A)\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\langle X\rangle &= \frac{\langle a^*|X|a\rangle}{\langle a^*|a\rangle} \\
&= \frac{l}{\sqrt{2}}(a^* + a) = \sqrt{2}\; l\;\; \mathrm{Re}(a)\,, \\
\langle P\rangle &= \frac{\langle a^*|P|a\rangle}{\langle a^*|a\rangle} \\
&= \frac{\hbar/l}{\sqrt{2}}(\mathrm{i}a^* - \mathrm{i}a) = \sqrt{2}\;\frac{\hbar}{l}\;\; \mathrm{Im}(a)\,.
\end{aligned}
$$

To compute the spread, we first compute X^2 and P^2 in terms of the ladder operators:

$$
\begin{aligned}
X^2 &= \frac{l^2}{2}(A^\dagger + A)^2 = \frac{l^2}{2}({A^\dagger}^2 + A^\dagger A + AA^\dagger + A^2) \\
&= \frac{l^2}{2}({A^\dagger}^2 + 2A^\dagger A + A^2 + 1)\,, \\
P^2 &= \frac{(\hbar/l)^2}{2}(\mathrm{i}A^\dagger - \mathrm{i}A)^2 = \frac{(\hbar/l)^2}{2}(-{A^\dagger}^2 + A^\dagger A + AA^\dagger - A^2) \\
&= \frac{(\hbar/l)^2}{2}(-{A^\dagger}^2 + 2A^\dagger A - A^2 + 1)\,.
\end{aligned}
$$

The computation of the spread follows

$$\begin{aligned}
\langle X^2 \rangle &= \frac{l^2}{2}(a^{*^2} + 2a^*a + a^2 + 1) = \frac{l^2}{2}(a^* + a)^2 + \frac{l^2}{2} \\
&= \langle X \rangle^2 + \frac{l^2}{2}\,, \text{ so that} \\
(\delta X)^2 &= \frac{l^2}{2} \quad \text{or} \quad \delta X = \frac{l}{\sqrt{2}}\,; \\
\langle P^2 \rangle &= \frac{(\hbar/l)^2}{2}(-a^{*^2} + 2a^*a - a^2 + 1) = \frac{(\hbar/l)^2}{2}(ia^* - ia)^2 + \frac{(\hbar/l)^2}{2} \\
&= \langle P \rangle^2 + \frac{(\hbar/l)^2}{2}\,, \text{ so that} \\
(\delta P)^2 &= \frac{(\hbar/l)^2}{2} \quad \text{or} \quad \delta P = \frac{\hbar/l}{\sqrt{2}}\,.
\end{aligned}$$

From the results above, we can see that $\delta X \delta P = \dfrac{\hbar}{2}$ as we would expect since the coherent states are the minimum uncertainty states.

Problem 2

A system is in an eigenstate of $\vec{L}^2$ with eigenvalue $2\hbar^2$. Since we know

$$\vec{L}^2|l,m\rangle = |l,m\rangle\hbar^2 l(l+1)\,,$$

it is obvious that

$$l = 1, \text{ and } m = -1, 0, +1\,.$$

To find out the action of $L_1L_2 + L_2L_1$, we consider the ladder operators for angular momentum:

$$\begin{aligned}
L_\pm &= L_1 \pm \mathrm{i}L_2\,, \\
L_\pm^2 &= L_1^2 - L_2^2 \pm \mathrm{i}(L_1L_2 + L_2L_1)\,.
\end{aligned}$$

So we have

$$L_1L_2 + L_2L_1 = \frac{1}{2\mathrm{i}}(L_+^2 - L_-^2)\,.$$

Now let us look at the action of these ladder operators acting on each of the eigenstate. When L_+ is applied, we have

$$
\begin{aligned}
L_+|l=1,m=1\rangle &= 0\,, \\
L_+|l=1,m=0\rangle &= |l=1,m=1\rangle\hbar\sqrt{(l-m)(l+m+1)} \\
&= |l=1,m=1\rangle\hbar\sqrt{2}\,, \\
L_+|l=1,m=-1\rangle &= |l=1,m=0\rangle\hbar\sqrt{2}\,,
\end{aligned}
$$

and when L_- is applied, we obtain similar expressions

$$
\begin{aligned}
L_-|l=1,m=1\rangle &= |l=1,m=0\rangle\hbar\sqrt{(l+m)(l-m+1)} \\
&= |l=1,m=0\rangle\hbar\sqrt{2}\,, \\
L_-|l=1,m=0\rangle &= |l=1,m=-1\rangle\hbar\sqrt{2}\,, \\
L_-|l=1,m=-1\rangle &= 0\,.
\end{aligned}
$$

Now we can compute the action of $L_1L_2+L_2L_1$,

$$
\begin{aligned}
\frac{1}{2\mathrm{i}}(L_+^2-L_-^2)|l=1,m=1\rangle &= |l=1,m=-1\rangle\frac{-1}{2\mathrm{i}}(\hbar\sqrt{2})^2\,, \\
\frac{1}{2\mathrm{i}}(L_+^2-L_-^2)|l=1,m=0\rangle &= 0 \text{ [eigenvalue is } 0]\,, \\
\frac{1}{2\mathrm{i}}(L_+^2-L_-^2)|l=1,m=-1\rangle &= |l=1,m=1\rangle\frac{1}{2\mathrm{i}}(\hbar\sqrt{2})^2\,.
\end{aligned}
$$

The first and last equations could be combined to give

$$
\begin{aligned}
&(L_1L_2+L_2L_1)\Big(|l=1,m=1\rangle,|l=1,m=-1\rangle\Big) \\
&= \Big(|l=1,m=-1\rangle\mathrm{i}\hbar^2,|l=1,m=1\rangle(-\mathrm{i}\hbar^2)\Big) \\
&= \Big(|l=1,m=1\rangle,|l=1,m=-1\rangle\Big)\begin{pmatrix} 0 & -\mathrm{i}\hbar^2 \\ \mathrm{i}\hbar^2 & 0 \end{pmatrix}
\end{aligned}
$$

and the eigenvalues of the matrix $\begin{pmatrix} 0 & -\mathrm{i}\hbar^2 \\ \mathrm{i}\hbar^2 & 0 \end{pmatrix}$ are $\pm\hbar^2$.

Problem 3

A perturbed harmonic oscillator has the Hamilton operator

$$H = \hbar\omega A^\dagger A + \mathrm{i}\hbar\Omega({A^\dagger}^2 - A^2) \quad \text{with } |\Omega| < \tfrac{1}{2}\omega\,.$$

Introduce new ladder operators B and $B^\dagger$ such that

$$B = \alpha A + \beta A^\dagger\,, \quad B^\dagger = \alpha^* A^\dagger + \beta^* A\,.$$

For 'good' ladder operators, we expect their commutator to be 1, so we have

$$\begin{aligned} 1 &= [B, B^\dagger] \\ &= [\alpha A + \beta A^\dagger, \alpha^* A^\dagger + \beta^* A] \\ &= |\alpha|^2 - |\beta|^2\,. \end{aligned}$$

Also, we could re-express the Hamiltonian in terms of the new ladder operator as $H = \hbar\omega' B^\dagger B + E_0$. Compute $B^\dagger B$,

$$\begin{aligned} B^\dagger B &= (\alpha A + \beta A^\dagger)(\alpha^* A^\dagger + \beta^* A) \\ &= |\alpha|^2 A^\dagger A + |\beta|^2 A A^\dagger + \alpha^*\beta {A^\dagger}^2 + \beta^*\alpha A^2 \\ &= (|\alpha|^2 + |\beta|^2) A^\dagger A + \alpha^*\beta {A^\dagger}^2 + \beta^*\alpha A^2 + |\beta|^2\,. \end{aligned}$$

Thus, we could compare terms in the expressions of H:

$$\begin{aligned} H &= \hbar\omega'(|\alpha|^2 + |\beta|^2) A^\dagger A + \hbar\omega'\alpha^*\beta {A^\dagger}^2 + \hbar\omega'\beta^*\alpha A^2 + \hbar\omega'|\beta|^2 + E_0 \\ &= \hbar\omega A^\dagger A + \mathrm{i}\hbar\Omega({A^\dagger}^2 - A^2)\,, \end{aligned}$$

giving rise to

$$\begin{aligned} E_0 &= -\hbar\omega'|\beta|^2\,, \\ \omega &= \omega'(|\alpha|^2 + |\beta|^2)\,, \\ \mathrm{i}\Omega &= \omega'\alpha^*\beta\,, \\ -\mathrm{i}\Omega &= \omega'\beta^*\alpha\,, \end{aligned}$$

where the last two equations are complex conjugate of each other and thus essentially the same.

All together, we have the following three equations:

$$|\alpha|^2 - |\beta|^2 = 1\,, \tag{1}$$

$$|\alpha|^2 + |\beta|^2 = \omega/\omega'\,, \tag{2}$$

$$|\alpha|\,|\beta| = |\Omega|/\omega'\,. \tag{3}$$

Since we know that

$$\begin{aligned}
|\alpha|^2 + |\beta|^2 \pm 2|\alpha|\ |\beta| &= (|\alpha| \pm |\beta|)^2\,, \\
|\alpha| \pm |\beta| &= \sqrt{|\alpha|^2 + |\beta|^2 \pm 2|\alpha|\ |\beta|}\,, \\
&= \sqrt{\frac{\omega}{\omega'} \pm \frac{2|\Omega|}{\omega'}}\,,
\end{aligned}$$

we could always solve for $|\alpha|$ and $|\beta|$ separately. This is not necessary, however, as what we want to obtain is the expression for the energy E_0 which is in terms of $|\beta|^2$ and ω'. Thus we consider the following:

$$\begin{aligned}
&|\alpha|^2 - |\beta|^2 = (|\alpha| + |\beta|)(|\alpha| - |\beta|) = \sqrt{\left(\frac{\omega}{\omega'}\right)^2 - \left(\frac{2\Omega}{\omega'}\right)^2} = 1\,, \\
&\text{so that} \quad \omega' = \sqrt{\omega^2 - (2\Omega)^2} = \sqrt{\omega^2 - 4\Omega^2}\,.
\end{aligned}$$

Also, from Eqs. (1),(2), we have

$$|\beta|^2 = \frac{1}{2}\frac{\omega}{\omega'} - \frac{1}{2}\,.$$

Therefore, the energy E_0 is given by

$$\begin{aligned}
E_0 &= -\hbar\omega'|\beta|^2 \\
&= -\hbar\omega'\ \frac{1}{2}\left(\frac{\omega}{\omega'} - 1\right) \\
&= -\hbar\ \frac{\omega - \omega'}{2} \\
&= -\hbar\ \frac{\omega - \sqrt{\omega^2 - 4\Omega^2}}{2}\,.
\end{aligned}$$

Problem 4

Given that the ground state energy E_0 of the Hamilton operator

$$H = \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 + F|X| \quad \text{with } M > 0,\ \omega > 0,\ F \text{ arbitrary}$$

is a function of the parameters M, ω, and F.

By the Hellmann-Feynman Theorem, we have

$$\frac{\partial E_0}{\partial F}\bigg|_{F=0} = \left\langle \frac{\partial E_0}{\partial F} \right\rangle\bigg|_{F=0} = \langle |X| \rangle\big|_{F=0}\,.$$

When $F = 0$, the Hamilton operator is that of a Harmonic Oscillator, so that the ground state wave function is simply

$$\psi_0(x) = \pi^{-1/4} l^{-1/2} \exp\left[-\frac{1}{2}\left(\frac{x}{l}\right)^2 \right] \quad \text{with} \quad l = \sqrt{\frac{\hbar}{M\omega}}\,.$$

Therefore, the computation follows:

$$\begin{aligned}
\frac{\partial E_0}{\partial F}\bigg|_{F=0} &= \int dx\ |x|\ \frac{1}{\sqrt{\pi}}\ \frac{1}{l}\ e^{-(x/l)^2} \\
&= \frac{2}{\sqrt{\pi}}\ \frac{1}{l} \int_0^\infty dx\ x e^{-(x/l)^2} \\
&= \frac{2}{\sqrt{\pi}}\ \frac{1}{l} \int_0^\infty dx\ \frac{d}{dx}\left(-\frac{1}{2} l^2 e^{-(x/l)^2} \right) \\
&= \frac{2}{\sqrt{\pi}}\ \frac{1}{l}\ \frac{1}{2} l^2 \\
&= \frac{l}{\sqrt{\pi}} \\
&= \sqrt{\frac{\hbar}{\pi M\omega}}\,.
\end{aligned}$$