1. A particle (mass M, position operator X, momentum operator P) moves along the x axis under the influence of the Hamilton operator $H = \frac{1}{2M}(P - M\omega X)^2$ where $\omega > 0$ is a constant frequency parameter.

(a) State the Heisenberg equations of motion for $P(t)$ and $X(t)$, and solve them. Then evaluate the commutator $[X(t), X(t_0)]$. [9 marks]

(b) Express $P(t)$, $P(t_0)$, $P(t) - M\omega X(t)$, and H in terms of $X(t)$ and $X(t_0)$. [6 marks]

(c) Find the time transformation function $\langle x, t | x', t_0 \rangle$ by first establishing its derivatives with respect to x, x', and $T = t - t_0$. [10 marks]

2. A and $A^\dagger$ are the ladder operators of a harmonic oscillator. A hermitian operator Z is such that

$$ZA^\dagger = (1 - \lambda)A^\dagger Z \quad \text{with } 0 < \lambda < 1,$$

and is normalized to unit trace, $\operatorname{tr}\{Z\} = 1$.

(a) Determine the normally ordered form of Z. [15 marks]

(b) Show that Z commutes with $A^\dagger A$. Then express Z as a function of $A^\dagger A$. [10 marks]

3. Orbital angular momentum vector $\vec{L}$ with cartesian components L_1, L_2, and L_3. The system is in an eigenstate of $\vec{L}^2$ with eigenvalue $6\hbar^2$.

(a) What are the possible outcomes when one measures (i) L_1^2; (ii) L_2^2; (iii) $L_1^2 + L_2^2$? [6 marks]

(b) What are the possible outcomes when one measures $L_1^2 - L_2^2$? [12 marks]

(c) What are the expectation values and the spreads of L_1 and L_2 in an eigenstate of L_3 with eigenvalue $m\hbar$? [7 marks]

4. A harmonic oscillator (mass M, natural frequency ω, position operator X, momentum operator P) is perturbed by a δ-function potential of strength $\propto V$, so that the Hamilton operator is

$$H = H_0 + H_1 \quad \text{with} \quad H_0 = \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 \quad \text{and} \quad H_1 = V\sqrt{\frac{\hbar}{M\omega}}\,\delta(X)\,,$$

where $\delta(X) = \Big(|x\rangle\langle x|\Big)\Big|_{x=0}$. As usual, we denote the eigenkets of H_0 by $|n\rangle$ with $n = 0, 1, 2, \ldots$.

(a) Determine the $\xi = 0$ value of the nth Hermite polynomial $H_n(\xi)$ with the aid of the generating function

$$\mathrm{e}^{2t\xi - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\xi)\,.$$

Then find $\langle x|n\rangle\Big|_{x=0}$ for $n = 0, 1, 2, \ldots$. [10 marks]

(b) Write $E_n(V)$ for the V-dependent nth eigenvalue of H and determine

$$\left.\frac{\partial E_n}{\partial V}\right|_{V=0}.$$

for $n = 0, 1, 2, \ldots$. [10 marks]

(c) Use the large-m approximation $\begin{pmatrix} 2m \\ m \end{pmatrix} \simeq \dfrac{4^m}{\sqrt{\pi m}}$ to establish a large-n approximation for $\left.\dfrac{\partial E_n}{\partial V}\right|_{V=0}$. [5 marks]