1. **Liouville equation. Calculus of variations.**
   Remark: Parts (a) and (b) are not related to each other.

   (a) A particle of mass m moves along the x axis whereby the dynamics is governed by the Hamilton function
   $$H = \frac{p^2}{2m} - \gamma x p \quad \text{with } \gamma = \text{const}\,.$$
   (i) State the Liouville equation obeyed by the phase-space density $\varrho(t,x,p)$.
   (ii) Solve this quasilinear partial differential equation to determine $\varrho(t,x,p)$ in terms of $\varrho_0(x,p) = \varrho(t=0,x,p)$.
   (iii) Verify that
   $$\int \mathrm{d}x\,\mathrm{d}p\,\varrho(t,x,p) = \int \mathrm{d}x\,\mathrm{d}p\,\varrho_0(x,p)$$
   where the integration covers all of phase space. [15 marks]

   (b) What is the smallest value that you can get for
   $$\int_0^1 \mathrm{d}x\,x\left[\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right]^2$$
   if the permissible $y(x)$ are restricted by $\displaystyle\int_0^1 \mathrm{d}x\,xy(x) = 1$ as well as $y(1) = 0$ and $\displaystyle\frac{\mathrm{d}y}{\mathrm{d}x}(0) = 0$? [10 marks]

2. **Group theory.** The elements $\boldsymbol{g} = (a,b,u)$ of G are ordered triplets of two real numbers a, b, and a complex phase factor u, one element for each triplet, whereby no restrictions are imposed on a, b, or u other than $|u| = 1$. Their compositions, written as products, are defined by the rule
   $$\boldsymbol{g}_1\boldsymbol{g}_2 = \left(a_1 + a_2, b_1 + b_2, u_1 u_2 \mathrm{e}^{\mathrm{i}a_1 b_2}\right) \quad \text{for} \quad \boldsymbol{g}_1 = (a_1,b_1,u_1)\,, \boldsymbol{g}_2 = (a_2,b_2,u_2)\,.$$

   (a) Show that, with this composition rule, G is a group. In particular, state the triplets for the neutral element $\boldsymbol{e}$ and the inverse $\boldsymbol{g}^{-1}$ of $\boldsymbol{g}$. [8 marks]

   (b) For each period $N = 2,3,4,\ldots$, group G has a unique cyclic subgroup G_N. What are the elements of G_N? [5 marks]

   (c) Which relation among a_1, b_1, u_1 and a_2, b_2, u_2 is implied by $\boldsymbol{g}_1\boldsymbol{g}_2 = \boldsymbol{g}_2\boldsymbol{g}_1$? [5 marks]

   (d) Given group element $\boldsymbol{a} = (a_0, 0, 1)$ with $a_0 > 0$, find $\boldsymbol{b} = (0, b_0, 1)$ with the smallest $b_0 > 0$ such that $\boldsymbol{ab} = \boldsymbol{ba}$. Which elements of G constitute the smallest Abelian subgroup that contains both $\boldsymbol{a}$ and $\boldsymbol{b}$? [7 marks]

3. Laplace transforms.

Remark: Part (c) is not related to parts (a) and (b).

(a) Function $f(t)$ obeys the linear differential equation

$$\left(t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\frac{\mathrm{d}}{\mathrm{d}t} + t\right) f(t) = g(t) \qquad \text{with} \quad \frac{\mathrm{d}f}{\mathrm{d}t}(0) = \frac{1}{2}g(0)$$

where $g(t)$ is a given inhomogeneity.

(i) Derive the differential equation obeyed by the Laplace transform $F(s)$ of $f(t)$. Hint: It will involve $f(0)$ and the Laplace transform $G(s)$ of $g(t)$.

(ii) Then state $f(t)$ for given $f(0)$ and $g(t)$. [10 marks]

(b) (i) What is $f(t)$ for $f(0) = 0$ and $g(t) = 1$?

(ii) What is $f(t)$ for $f(0) = -2$ and $g(t) = 3\sin(2t)$? [5 marks]

(c) Evaluate the integral

$$\int_0^\infty \mathrm{d}t\, \frac{J_0(at) - J_0(bt)}{t} \qquad (a, b > 0)$$

with the aid of a Laplace transform. — Note: The Laplace transform of the Bessel function $J_0(t)$ is $1\big/\sqrt{1+s^2}$. [10 marks]

4. Functions of a complex variable. The function $f(z)$ is analytic everywhere in the complex plane except for a few isolated point singularities. One of its Laurent series is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \qquad \text{with} \quad a_n = \begin{cases} (1/3)^{n+1} & \text{for } n \geq 0\,, \\ -3(-1)^{n/2} & \text{for } n < 0 \text{ even}\,, \\ (-1)^{(n+1)/2} & \text{for } n < 0 \text{ odd}\,, \end{cases}$$

which applies for $1 < |z| < 3$.

(a) Find a simple, compact expression for $f(z)$ that is valid everywhere except at the singularities. Identify the singularities and determine their residues. [10 marks]

(b) What are the coefficients in the Taylor series of $f(z)$ at $z = 0$? For which z does this Taylor series converge? [5 marks]

(c) Integrate $f(z)$ along the following closed path: From $z = 0$ proceed just below the real axis to $z = \infty$, then return to $z = 0$ just above the real axis. [5 marks]

(d) Integrate $f(z)$ along the following closed path composed of four straight lines: First from $z_1 = -2(1 + \mathrm{i})$ to $z_2 = 2(1 + \mathrm{i})$, then to $z_3 = -2(1 - \mathrm{i})$, then to $z_4 = 2(1 - \mathrm{i})$, then back to z_1. [5 marks]