Problem 1 (20=10+10 points)
As in lecture, operators U and V are the standard complementary pair of cyclic unitary operators of period N for an N-dimensional quantum degree of freedom, and their eigenkets are denoted by $|u_k\rangle$ and $|v_l\rangle$, respectively, for $k, l = 1, 2, \ldots, N$.

(a) For $F = \sum_{k,l=1}^{N} f_{kl} U^k V^l$ and $G = \sum_{k,l=1}^{N} g_{kl} U^k V^l$, express the trace $\mathrm{tr}\left\{F^\dagger G\right\}$ in terms of the complex coefficients f_{kl} and g_{kl}.

(b) Show that $\sum_{k=1}^{N} \langle u_k| = \sqrt{N}\langle v_N|$. By analogy, what is $\sum_{l=1}^{N} |v_l\rangle$?

Problem 2 (40=5+15+10+10 points)
We consider two hermitian operators, Q and Γ. The eigenvalues q of operator Q are all positive numbers: $0 < q < \infty$; the eigenvalues γ of operator Γ are all real numbers: $-\infty < \gamma < \infty$; and their eigenstates are related by

$$\langle q|\gamma\rangle = \frac{1}{\sqrt{2\pi}} q^{\mathrm{i}\gamma} \,,$$

whereby

$$\langle q|q'\rangle = q\delta(q - q') \,, \qquad \langle \gamma|\gamma'\rangle = \delta(\gamma - \gamma')$$

are the respective orthonormality statements.

(a) Explain why Q and Γ constitute a pair of complementary observables.

(b) Evaluate $\int_0^\infty \frac{\mathrm{d}q}{q} \langle\gamma|q\rangle\langle q|\gamma'\rangle$ and $\int_{-\infty}^{\infty} \mathrm{d}\gamma \, \langle q|\gamma\rangle\langle\gamma|q'\rangle$ to establish the completeness relations

$$\int_0^\infty \frac{\mathrm{d}q}{q} |q\rangle\langle q| = 1 \,, \qquad \int_{-\infty}^{\infty} \mathrm{d}\gamma \, |\gamma\rangle\langle\gamma| = 1 \,.$$

(c) Show that $Q^{\mathrm{i}\gamma'}|\gamma\rangle = |\gamma + \gamma'\rangle$ for all real numbers γ and γ'.

(d) Use this to demonstrate that

$$\mathrm{e}^{\mathrm{i}\beta\Gamma} Q^{\mathrm{i}\gamma} = \mathrm{e}^{\mathrm{i}\beta\gamma} Q^{\mathrm{i}\gamma} \mathrm{e}^{\mathrm{i}\beta\Gamma}$$

for all real numbers β and γ.

Problem 3 (15=5+10 points)

(a) First show that

$$\mathrm{e}^{-\epsilon B}\mathrm{e}^{A}\mathrm{e}^{\epsilon B} = \mathrm{e}^{\mathrm{e}^{-\epsilon B}\,A\,\mathrm{e}^{\epsilon B}}$$

where A and B are operators and ϵ is a complex number.

(b) Then demonstrate that

$$\left[\mathrm{e}^{A}, B\right] = \int_0^1 \mathrm{d}x\;\mathrm{e}^{(1-x)A}\,[A, B]\,\mathrm{e}^{xA}\,.$$

Problem 4 (25=15+10 points)

All eigenvalues of the hermitian operator A are positive.

(a) Verify that

$$\log A = \int_0^\infty \mathrm{d}\alpha \left(\frac{1}{\alpha+1} - \frac{1}{\alpha+A}\right) = \int_0^\infty \mathrm{d}\beta\,\frac{\mathrm{e}^{-\beta} - \mathrm{e}^{-\beta A}}{\beta}$$

are two valid integral representations of $\log A$.

(b) Consider an infinitesimal variation δA and establish

$$\delta \log A = \int_0^\infty \mathrm{d}\alpha\,\frac{1}{\alpha+A}\,\delta A\,\frac{1}{\alpha+A}\,.$$