Problem 1 (10 marks)
Position operator X, momentum operator P. Simplify $PX^3P - XPXPX$.

Problem 2 (5+5=10 marks)
Consider functions $f(X) = \int \mathrm{d}x\,|x\rangle f(x)\langle x|$ of position operator X.

(a) Which property must $f(x)$ have, if $f(X)$ is hermitian?

(b) Which property must $f(x)$ have, if $f(X)$ is unitary?

Problem 3 (10 marks)
Position operator X, momentum operator P; real numerical parameters x and p. How are x and p related to each other if

$$\mathrm{e}^{\mathrm{i}pX/\hbar}\mathrm{e}^{\mathrm{i}xP/\hbar} = \mathrm{e}^{\mathrm{i}xP/\hbar}\mathrm{e}^{\mathrm{i}pX/\hbar}$$

holds?

Problem 4 (10+8+8+9=35 marks)
Operator U is defined by its position matrix elements

$$\langle x|U|x'\rangle = \frac{1}{\sqrt{2\pi}\,a}\mathrm{e}^{\mathrm{i}xx'/a^2}$$

where $a > 0$ is a numerical length parameter.

(a) Show that U is unitary.

(b) Determine the mixed position-momentum matrix elements $\langle x|U|p'\rangle$.

(c) Determine the momentum matrix elements $\langle p|U|p'\rangle$.

(d) What is $\langle x|U^2$?

Problem 5 (10+15+10=35 marks)
State ket $|\ \rangle$ is specified by its position wave function

$$\psi(x) = \langle x|\ \rangle = \sqrt{\kappa}\,\mathrm{e}^{-\kappa|x|} \qquad \text{with parameter } \kappa > 0\,.$$

(a) Determine the expectation value of the unitary operator $\mathrm{e}^{\mathrm{i}kX}$ where k is a real parameter.

(b) Express $\langle x|\mathrm{e}^{\frac{1}{2}\mathrm{i}aP/\hbar}|\ \rangle$ and $\langle\ |\mathrm{e}^{\frac{1}{2}\mathrm{i}aP/\hbar}|x\rangle$ in terms of $\psi(x)$ whereby a is a real parameter; then determine the expectation value of $\mathrm{e}^{\mathrm{i}aP/\hbar}$.

(c) Extract $\langle X\rangle$ and $\langle X^2\rangle$ out of $\langle \mathrm{e}^{\mathrm{i}kX}\rangle$, as well as $\langle P\rangle$ and $\langle P^2\rangle$ out of $\langle \mathrm{e}^{\mathrm{i}aP/\hbar}\rangle$. Then determine the position spread δX and the momentum spread δP and verify that they obey the uncertainty relation.