1. **Quantum kinematics.** Consider motion along the x axis with, as usual, position operator X and momentum operator P as well as their respective eigenkets $|x\rangle$ and $|p\rangle$.

   (a) Show that the operator defined by

$$U = \int_{-\infty}^{\infty} \mathrm{d}\lambda \, |p = \lambda p_0\rangle \sqrt{p_0 x_0} \langle x = \lambda x_0|$$

   is unitary, whereby the parameter $x_0 > 0$ is a reference length and $p_0 > 0$ is a reference momentum. [8 marks]

   (b) Show that U turns position kets into momentum kets in accordance with

$$U|x\rangle = |p = p_0 x/x_0\rangle \sqrt{p_0/x_0} \, .$$

   [6 marks]

   (c) Conversely, do you get a position bra if U is applied to $\langle p|$? Justify your answer. [3 marks]

   (d) Show that

$$U^\dagger f(X, P) U = f(-x_0 P/p_0, p_0 X/x_0)$$

   for any function of X and P. [8 marks]

2. **Temporal evolution.** The Hamilton operator of a two-dimensional system is

$$H = \frac{1}{2M}(P_1^2 + P_2^2) + \omega(X_1 P_2 - X_2 P_1)$$

   with constant mass M and frequency ω, and $[X_j, P_k] = \mathrm{i}\hbar\delta_{jk}$ for $j, k = 1, 2$.

   (a) State the Heisenberg equations of motion for X_1, X_2, P_1, and P_2. [4 marks]

   (b) Show that $P_1^2 + P_2^2$ and $X_1 P_2 - X_2 P_1$ are constants of motion. [6 marks]

   (c) Solve the equations of motion of part (a) for $P_1(t)$ and $P_2(t)$. [7 marks]

   (d) Determine the time transformation function $\langle x_1, x_2, t|p_1, p_2, t_0\rangle$. [8 marks]

3. Orbital angular momentum. As usual we denote by L_1, L_2, and L_3 the cartesian components of the orbital angular momentum vector operator $\vec{L}$, and the common eigenkets of $\vec{L}^2$ and L_3 by $|l, m\rangle$. The state of the system is given by a ket of the form

$$|\,\rangle = |l = 1, m = 1\rangle\alpha + |l = 1, m = -1\rangle\beta\,,$$

where α and β are complex coefficients with $|\alpha|^2 + |\beta|^2 = 1$.

(a) Determine the expectation values of L_1, L_2, and L_3 as well as their spreads δL_1, δL_2, and δL_3. [10 marks]

(b) For each pair of the spreads δL_1, δL_2, and δL_3, state the uncertainty relation that the pair obeys. [5 marks]

(c) Verify that the equal sign applies in the uncertainty relation for δL_1 and δL_2 if $\alpha = \dfrac{3}{5}$ and $\beta = \dfrac{4}{5}$. [5 marks]

(d) What can you say, quite generally, about the coefficients α and β if the equal sign applies in the uncertainty relation for δL_1 and δL_2? [5 marks]

4. Perturbed oscillator. A harmonic oscillator (ladder operators A, $A^\dagger$; circular frequency ω) is perturbed by a cubic interaction of strength $\hbar\Omega$,

$$H = H_0 + H_1 \qquad \text{with} \qquad H_0 = \hbar\omega A^\dagger A\,, \quad H_1 = \hbar\Omega(A^\dagger A A^\dagger + A A^\dagger A)\,.$$

(a) Determine the 1st-order change of the nth energy level. [5 marks]

(b) Determine the 2nd-order change of the nth energy level. [10 marks]

(c) What is the energy spacing ΔE_n between the nth and the $(n-1)$th level when the perturbation H_1 is taken into account up to 2nd order? [5 marks]

(d) For which values of n will it surely be necessary to include higher than 2nd-order corrections? [5 marks]