Problem 1 (25 marks)
Consider the set G whose elements are the complex 2×2 matrices M that obey

$$M^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Show that G is a group, with matrix multiplication as the group composition. Then define G_+ as a subset of G such that $\det\{M\} = 1$ for all $M \in G_+$, and show that G_+ is a subgroup of G. Give an example for a group element $M \in G$ that is <u>not</u> in G_+.

Problem 2 (25 marks)
The elements of G_+ are the matrices $M(a,b,c,d) = \begin{pmatrix} a & \mathrm{i}b \\ \mathrm{i}c & d \end{pmatrix}$ with the real parameters a, b, c, d subject to $ad + bc = 1$. Which of the restrictions

(a) $b = c$ and $a = d\,$, (b) $b = 0\,$, (c) $c = 0\,$, (d) $M^\dagger = M$

defines a subgroup? Are there Abelian subgroups among them?

Problem 3 (25 marks)
Use Laplace-transformation techniques to evaluate

$$\int_0^t \mathrm{d}\tau\, (t-\tau)^m\, \tau^n$$

for $m, n = 0, 1, 2, 3, \dots$.

Problem 4 (25 marks)
The given hermitian $n \times n$ matrix S is known to be a bit larger than the $n \times n$ identity matrix E, in the sense that their difference $S - E$ has small positive eigenvalues. We want to calculate a $n \times n$ matrix T such that $T^\dagger S T = E$ by an iteration of the form

$$T_0 = E\,, \qquad T_{k+1} = T_k + \lambda T_k (T_k^\dagger S T_k - E) \quad \text{for } k = 0, 1, 2, \dots\,,$$

where λ is a complex parameter that we wish to choose optimally. Determine the best choice for λ by the following strategy. First express $\epsilon_{k+1} = T_{k+1}^\dagger S T_{k+1} - E$ as a cubic polynomial in $\epsilon_k = T_k^\dagger S T_k - E$, that is

$$\epsilon_{k+1} = c_1 \epsilon_k + c_2 \epsilon_k^2 + c_3 \epsilon_k^3\,,$$

and then choose λ such that c_1 vanishes and $|c_2|$ is as small as possible. For your choice of λ, now find ϵ_0, ϵ_1, and ϵ_2. — Express S^{-1} in terms of T.