1. Calculus of Variations (15 marks)

For given $a > 0$, what is the smallest value that you can get for

$$\int_{-a}^{a} \mathrm{d}x \left[\frac{\mathrm{d}}{\mathrm{d}x} y(x)\right]^2$$

if the permissible $y(x)$ are restricted by

$$y(\pm a) = 0 \qquad \text{and} \qquad \int_{-a}^{a} \mathrm{d}x\, |x|\, y(x) = a^3\,?$$

2. Group Theory (35=15+15+5 marks)

The elements $\boldsymbol{g} = (a, b)$ of the set G are ordered pairs of complex numbers, subject to $|a|^2 - |b|^2 = 1$, one element of G for each a, b pair. The composition of two elements of G, written as a product, is defined by

$$\boldsymbol{g}_1\boldsymbol{g}_2 = (a_1a_2 + b_1^*b_2, b_1a_2 + a_1^*b_2) \quad \text{for} \quad \boldsymbol{g}_1 = (a_1, b_1)\,, \boldsymbol{g}_2 = (a_2, b_2)\,.$$

(a) Show that, with this composition rule, G is a group. In particular, state the a, b pairs for the neutral element $\boldsymbol{e}$ and the inverse $\boldsymbol{g}^{-1}$ of $\boldsymbol{g}$.

(b) Consider the following four restrictions on the values of a and b:

$$\begin{array}{rl} \text{(i)} & \mathrm{Im}(a) = \mathrm{Im}(b)\,; \\ \text{(ii)} & \mathrm{Im}(a) = -\mathrm{Im}(b)\,; \\ \text{(iii)} & \mathrm{Im}(b) = 0\,; \\ \text{(iv)} & b = 0\,. \end{array}$$

Which of them define subgroups of G?

(c) Of those restrictions that define subgroups, which ones define abelian subgroups?

3. Laplace Transform (15 marks)

For $T > 0$, find the Laplace transform $F(s)$ of

$$f(t) = \begin{cases} 1 & \text{if} \quad \sin(2\pi t/T) > 0\,, \\ -1 & \text{if} \quad \sin(2\pi t/T) < 0\,. \end{cases}$$

4. Contour Integration and Laplace Transformation (35=10+15+10 marks)

The functions $b_n(t)$ with $n = 0, \pm 1, \pm 2 \ldots$ are defined by their generating function in accordance with

$$\mathrm{e}^{\frac{1}{2}t(z-1/z)} = \sum_{n=-\infty}^{\infty} z^n b_n(t) = \sum_{n=-\infty}^{\infty} (-z)^{-n} b_n(t)\,.$$

(a) Show that

$$(-1)^n b_n(t) = b_{-n}(t) = \int_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi\mathrm{i}z}\, z^n\, \mathrm{e}^{\frac{1}{2}t(z-1/z)}\,,$$

where the contour $\mathcal{C}$ circles $z = 0$ once counter-clockwise along the unit circle.

(b) Use this to calculate the Laplace transforms $B_n(s)$ of all $b_n(t)$.

(c) Evaluate the integrals $\int_0^\infty \mathrm{d}t\, b_0(t)$ and $\int_0^\infty \mathrm{d}t\, \frac{b_1(t)}{t}$.

Hint: Do not worry about interchanging the order of summations and integrations.