Problem 1 (25 marks)
What is the smallest value that you can get for

$$\int_0^\infty \mathrm{d}x \left(\left[\frac{\mathrm{d}}{\mathrm{d}x} y(x) \right]^2 + [y(x)]^3 \right)$$

if the permissible $y(x)$ are such that $y(0) = 1$ and $y(x) \to 0$ for $x \to \infty$?

Problem 2 (30 marks)
Mass m is moving without friction on the surface specified by $z = \sqrt{x^2 + y^2 + a^2}$ with $a > 0$, while the gravitational force $m\boldsymbol{g} = -mg\boldsymbol{e}_3 \mathrel{\widehat{=}} (0, 0, -mg)$ is acting. Find the Lagrange function $L(\zeta, \varphi, \dot{\zeta}, \dot{\varphi})$ where x and y are related to the coordinates ζ, φ by

$$x = a \sinh\zeta \, \cos\varphi \,, \qquad y = a \sinh\zeta \, \sin\varphi$$

with $\zeta \geq 0$. Then determine the corresponding Hamilton function and state the Hamilton equations of motion.

Problem 3 (15 marks)
The phase space density $\rho(x, p, t)$ obeys the Liouville equation to a certain Hamilton function $H(x, p, t)$. Show that

$$\int \mathrm{d}x \, \mathrm{d}p \, \rho(x, p, t)$$

does not depend on time t, whereby the integration covers the whole phase space.

Problem 4 (30 marks)
Function $\mathsf{z}(x, y)$ obeys the quasi-linear partial differential equation (qlPDE)

$$\left[(x + \mathsf{z}) \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial y} \right] \mathsf{z} = 0 \,.$$

Determine the solution of this qlPDE for $\mathsf{z}(x, 0) = x$.