1. Calculus of Variations (15 marks)

What is the smallest value you can get for the integral

$$\int_1^\infty \mathrm{d}x \left[\left(\frac{\mathrm{d}}{\mathrm{d}x} f(x) \right)^2 + \frac{5 f(x)^3}{2\sqrt{x}} \right]$$

if $f(x)$ is restricted by $f(1) = 1$ and $f(x) \to 0$ for $x \to \infty$? — Hint: If you should need to solve a 2nd-order differential equation, it would have a very simple solution that you would be able to guess.

2. Group Theory (30=15+10+5 marks)

The eight elements of group G are mappings of complex numbers, among them E, A, and B, which are given by

$$E: \ z \mapsto E(z) = z\,, \qquad A: \ z \mapsto A(z) = z^*\,, \qquad B: \ z \mapsto B(z) = \mathrm{i}z\,.$$

The group composition law is illustrated by

$$AB: \ z \mapsto A(B(z)) = (\mathrm{i}z)^* = -\mathrm{i}z^*\,, \qquad BA: \ z \mapsto B((A(z)) = \mathrm{i}z^*\,.$$

(a) Complete the group composition table:

	E	A	B	B^2	B^3	AB	AB^2	AB^3
E	E	A	B	B^2	B^3	AB	AB^2	AB^3
A	A		AB	AB^2	AB^3			
B	B		B^2	B^3				
B^2	B^2		B^3					
B^3	B^3							
AB	AB		AB^2	AB^3				
AB^2	AB^2		AB^3					
AB^3	AB^3							

(b) Give a complete list of all subgroups with two elements or four elements.

(c) Which of these subgroups are abelian?

3. Laplace Transform (20 marks)

Use Laplace transform techniques to find the function $f(t)$ that obeys

$$f(t) - t\frac{\mathrm{d}}{\mathrm{d}t}f(t) = 2\int_0^t \mathrm{d}t'\, f(t')f(t-t') \qquad \text{and} \qquad \int_0^\infty \frac{\mathrm{d}t}{t}\, f(t) = \pi\,.$$

4. Complex Calculus (35=7+10+10+8 marks)

In order to give a unique meaning to the function $f(z) = (z^2-1)^{\frac{1}{2}}$ of the complex variable $z = x + \mathrm{i}y$, we choose the cut along the real axis from $z = -1$ to $z = 1$ and define

$$\begin{aligned} f(z) = (z^2-1)^{\frac{1}{2}} &= \sinh\theta\cos\phi + \mathrm{i}\cosh\theta\sin\phi \\ \text{for} \quad z &= \cosh\theta\cos\phi + \mathrm{i}\sinh\theta\sin\phi \qquad \text{with} \quad \theta > 0\,, \end{aligned}$$

which is such that $f(z)/z \to 1$ for $|z| \to \infty$.

(a) Verify that $f(z)^2 + 1 = z^2$.

(b) For $z_0 = x_0$ with $|x_0| < 1$, what is the limit of $f(z_0 + \mathrm{i}\epsilon) - f(z_0 - \mathrm{i}\epsilon)$ for $0 < \epsilon \to 0$?

(c) Evaluate $\int_{\mathcal{C}} \mathrm{d}z\, f(z)$ where $\mathcal{C}$ is the closed curve that is obtained when ϕ covers the range from $\phi = 0$ to $\phi = 2\pi$ for a fixed value of θ.

(d) For $|z| > 1$ give an alternative definition of $f(z)$ in terms of its Laurent expansion around $z = 0$. Confirm that the residue has the value implied by the result of part (c).