Problem 1 (35=7+8+10+5+5 marks)
As in lecture, operators U and V are the standard complementary pair of cyclic unitary operators of period N for an N-dimensional quantum degree of freedom, and their eigenkets are denoted by $\left|u_k\right\rangle$ and $\left|v_l\right\rangle$, respectively, for $k,l=1,2,\ldots,N$, with $u_k=\mathrm{e}^{\mathrm{i}2\pi k/N}$ and $v_l=\mathrm{e}^{\mathrm{i}2\pi l/N}$.

(a) Show that $(UV)^N=(-1)^{N-1}$.

(b) More generally, what do you get for $\bigl(U^mV^n\bigr)^N$?

(c) Unitary operator S is defined by $S\left|v_k\right\rangle=\left|u_k\right\rangle$ for $k=1,2,\ldots,N$. Find

$$\left\langle u_k\right|S\,,\quad S\left|u_k\right\rangle,\quad \text{and}\quad \left\langle v_k\right|S\,.$$

(d) Show that $US=SV$.

(e) Show that S is a cyclic operator. What is its period?
Hint: Consider $N=2$ and $N>2$ separately.

Problem 2 (25=10+15 marks)
Mass M moves along the x axis whereby the Hamilton operator

$$H=v|P|-FX\quad \text{with constant } v \text{ and constant } F$$

governs the evolution. The time transformation function $\left\langle x,t_1\middle|p,t_2\right\rangle$ depends on the size of the velocity v and the strength of the force F.

(a) Use the Schrödinger equation to find $\left\langle x,t_1\middle|p,t_2\right\rangle$ in the case of $v=0$.

(b) Use the quantum action principle to determine the v dependence of $\left\langle x,t_1\middle|p,t_2\right\rangle$ and thus obtain this time transformation function for arbitrary values of v and F.
Hint: $\dfrac{\mathrm{d}}{\mathrm{d}y}\bigl(y|y|\bigr)=2|y|$.

Problem 3 (40=15+10+7+8 marks)
We consider two hermitian operators, Θ and Λ. The eigenvalues ϑ of operator Θ are in the range $0 < \vartheta < \pi$; the eigenvalues λ of operator Λ are all real numbers: $-\infty < \gamma < \infty$; and their eigenstates are related by

$$\langle\vartheta|\lambda\rangle = \frac{1}{\sqrt{2\pi}}\Bigl(\tan\frac{\vartheta}{2}\Bigr)^{\mathrm{i}\lambda},$$

whereby

$$\langle\vartheta|\vartheta'\rangle = \delta(\vartheta-\vartheta')\sin\vartheta\,, \qquad \langle\lambda|\lambda'\rangle = \delta(\lambda-\lambda')$$

are the respective orthonormality statements.

(a) Evaluate $\displaystyle\int_0^\pi \frac{\mathrm{d}\vartheta}{\sin\vartheta}\,\langle\lambda|\vartheta\rangle\langle\vartheta|\lambda'\rangle$ and $\displaystyle\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\langle\vartheta|\lambda\rangle\langle\lambda|\vartheta'\rangle$ to establish the completeness relations

$$\int_0^\pi \frac{\mathrm{d}\vartheta}{\sin\vartheta}\,|\vartheta\rangle\langle\vartheta| = 1\,, \qquad \int_{-\infty}^{\infty}\mathrm{d}\lambda\,|\lambda\rangle\langle\lambda| = 1\,.$$

(b) Show that

$$\Bigl(\tan\frac{\Theta}{2}\Bigr)^{\mathrm{i}\lambda'}|\lambda\rangle = |\lambda+\lambda'\rangle$$

for all real numbers λ and λ'.

(c) Use this to demonstrate that

$$\mathrm{e}^{\mathrm{i}\mu\Lambda}\Bigl(\tan\frac{\Theta}{2}\Bigr)^{\mathrm{i}\lambda} = \mathrm{e}^{\mathrm{i}\mu\lambda}\Bigl(\tan\frac{\Theta}{2}\Bigr)^{\mathrm{i}\lambda}\mathrm{e}^{\mathrm{i}\mu\Lambda}$$

for all real numbers μ and λ.

(d) How is ϑ' related to ϑ and μ in $\langle\vartheta'| = \langle\vartheta|\mathrm{e}^{\mathrm{i}\mu\Lambda}$?