1. **Radiation from a small system (35=20+15 marks)**
A thin charged ring of radius R is rotating with constant angular velocity ω about an axis that is perpendicular to the plane of the ring and goes through the center of the ring. For the ring in the x, y-plane and rotating about the z-axis, we have the charge and current densities

$$\rho(\vec{r}, t) = \frac{e}{2\pi R}\,\delta(z)\,\delta(s - R)\,f(\varphi - \omega t)\,, \quad \vec{j}(\vec{r}, t) = R\omega\rho(\vec{r}, t)\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix},$$

where $(x, y, z) = (s\cos\varphi, s\sin\varphi, z)$ with $s > 0$, and $f(\varphi) = f(\varphi + 2\pi)$ is a periodic function of the azimuth φ.

(a) Treating this charge distribution as a small system, consider $f(\varphi) = \cos\varphi$ and determine first the angular distribution of the radiated power, averaged over one period of the circular motion, by taking into account electric dipole radiation, magnetic dipole radiation, and electric quadrupole radiation. Then find the total radiated power.

(b) Repeat for $f(\varphi) = \cos(2\varphi)$.

2. **Radiation from a large system (35=25+10 marks)**
Two half-wave antennas (length $L = \frac{1}{2}\lambda = \pi c/\omega$) are parallel to the z-axis at a distance $a > 0$, with their centers at $x = \pm\frac{1}{2}a$, so that the electric current density is given by $\vec{j}(\vec{r}, t) = \vec{j}_+(\vec{r}, t) + \vec{j}_-(\vec{r}, t)$ with

$$\vec{j}_\pm(\vec{r}, t) = \begin{cases} \vec{e}_z I\cos(\omega t \mp \frac{1}{2}\beta)\,\delta(x \mp \frac{1}{2}a)\,\delta(y)\,\cos(\pi z/L) & \text{for } |z| < \frac{1}{2}L\,, \\ 0 & \text{for } |z| > \frac{1}{2}L\,, \end{cases}$$

where I is the current fed into the antennas at frequency ω, and β is the relative phase between the currents in the two antennas.

(a) Find the angular distribution of the radiated power, averaged over one period of the oscillation.

(b) Determine a and β such that the power radiated in the direction $\vec{n} = +\vec{e}_x$ is particularly large and the power radiated in direction $\vec{n} = -\vec{e}_x$ is particularly small.

3. Retarded time (15 marks)

A charge moves along the trajectory $\vec{R}(t)$, and its electromagnetic field, observed at position $\vec{r}$ at time t, was radiated by the charge at the retarded time $t_{\rm ret}$, when the charge was at $\vec{R}(t_{\rm ret})$, with $t_{\rm ret}$ specified by

$$t = t_{\rm ret} + \frac{1}{c}|\vec{r} - \vec{R}(t_{\rm ret})|\,,$$

where we regard $t_{\rm ret}$ as a function of $\vec{r}$ and t. Show that

$$\left(\vec{\nabla} t_{\rm ret}\right)^2 - \left(\frac{1}{c}\frac{\partial t_{\rm ret}}{\partial t}\right)^2 = 0\,.$$

4. Synchrotron radiation (15 marks)

Apply the relativistic version of Larmor's energy-loss formula to a charge e that moves with constant speed $v \lesssim c$ on a circular orbit of radius R, and thus re-derive the total radiated power of synchrotron radiation.