**1. Force between co-moving charges (25=8+9+8 marks)**
Electron 1 is moving along the $z$-axis with constant velocity $\vec{v} = v\,\vec{e}_z$, so that its trajectory is $\vec{r}_1(t) = \vec{v}t$. Electron 2 is co-moving at a fixed distance $\vec{a}$ from electron 1, so that its trajectory is given by $\vec{r}_2(t) = \vec{r}_1(t) + \vec{a}$.

**(a)** Upon denoting by $\vec{E}(\vec{r}, t)$ the electric field associated with electron 1, show — by a *very simple* argument — that the corresponding magnetic field $\vec{B}(\vec{r}, t)$ is given by $\vec{B} = \dfrac{\vec{v}}{c} \times \vec{E}$.

**(b)** Determine the force $\vec{F}$ on electron 2, and express your answer in terms of the parallel and perpendicular components of $\vec{a} = \vec{a}_{\|} + \vec{a}_{\perp}$ with $\vec{a}_{\|} \parallel \vec{v}$ and $\vec{a}_{\perp} \perp \vec{v}$.

**(c)** For both $\vec{a} = \vec{a}_{\perp}$ and $\vec{a} = \vec{a}_{\|}$ compare $\vec{F}$ with the force in the common rest frame of the two electrons.

**2. Čerenkov radiation (10 marks)**
A charged particle moves with speed $v$ along the axis of a dielectric cylinder. The speed is so large that Čerenkov radiation of wavelength $\lambda$ is emitted. What fraction of this radiation passes into the surrounding vacuum through the cylindrical surface of radius $R \gg \lambda$?

**3. Diffraction (15 marks)**
All of the $x, y$ plane is covered by a thin conducting sheet except for an annulus (a ring-shaped opening) whose borders are two concentric circles with radii $a$ and $b$, $0 < a < b$. A plane wave with wavelength $\lambda$ is normally incident from the $z < 0$ side. The wavelength is short in the sense of $\lambda \ll a$ and $\lambda \ll b - a$. Employ the usual approximations and find the differential diffraction cross section.

**4. Free-electron laser (50=10+25+10+5 marks)**

In a free-electron laser (FEL), electrons (mass $m$, charge $e$) are injected at ultra-high speed into a helical magnetic field, which we will approximate by

$$\vec{B}(\vec{r}) = \begin{pmatrix} B\cos(k_0 z) \\ B\sin(k_0 z) \\ 0 \end{pmatrix} \quad \text{for} \quad 0 < z < L = \frac{2\pi}{k_0} N$$

and $\vec{B} = 0$ for $z < 0$ and $z > L$. The winding number $N$ is a large integer. We choose $\vec{r}(t) = \begin{pmatrix} v_\perp t \\ 0 \\ v_\parallel t \end{pmatrix}$ for $t < 0$ as the initial condition for an isolated electron, and take for granted that $v_\perp \ll v_\parallel \lesssim c$, as would be typical for FEL operation. Under these circumstances, FEL radiation is predominantly in the forward direction $\vec{n} = \vec{e}_z$.

**(a)** State the equations of motion and show that they are solved by

$$\vec{v}(t) = \begin{pmatrix} v_\perp \cos(k_0 z(t)) \\ v_\perp \sin(k_0 z(t)) \\ v_\parallel \end{pmatrix} \quad \text{for} \quad 0 < t < T = L/v_\parallel$$

in conjunction with $z(t) = v_\parallel t$, provided that the various constants $e$, $m$, $B$, $k_0$, $v_\perp$, $v_\parallel$, $v = \sqrt{v_\parallel^2 + v_\perp^2}$, $\gamma = 1/\sqrt{1-(v/c)^2}$ obey a certain relation.

**(b)** Calculate the angular-spectral distribution of the radiation in the forward direction by using

$$\frac{\mathrm{d}E(\omega)}{\mathrm{d}\Omega} = \frac{\omega^2}{4\pi^2 c^3} \left| \vec{n} \times \int_0^T \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\, \vec{\jmath}(\vec{k}, t) \right|^2 \quad \text{for} \quad \vec{k} = \frac{\omega}{c}\vec{n} = \frac{\omega}{c}\vec{e}_z\,,$$

where the time integral covers the period of acceleration.

**(c)** Determine the frequency $\omega_{\max}$ for which $\dfrac{\mathrm{d}E(\omega)}{\mathrm{d}\Omega}$ is maximal, and find this maximal value. How do these quantities depend on the winding number $N$?

**(d)** Find the smallest value of $\Delta\omega$ such that $\dfrac{\mathrm{d}E(\omega)}{\mathrm{d}\Omega} = 0$ for $\omega = \omega_{\max} \pm \Delta\omega$. How large is the fractional width $\dfrac{\Delta\omega}{\omega_{\max}}$ of the FEL frequency peak?