Problem 1 (20 marks)
Mr. Ah Beng is eyeing Miss Ah Lian, who sees a particle moving with velocity $\vec{u}$. If he took his eyes off her and watched the particle as well, which velocity $\vec{u}'$ would he observe, given that she has velocity $\vec{v}$ relative to him, with $\vec{v}$ perpendicular to $\vec{u}$? Verify that $u' \leq c$ for $v < c$ and $u \leq c$.

Problem 2 (20 marks)
In lecture we met the Lagrange density for the electromagnetic field,

$$\mathcal{L} = \frac{1}{4\pi}\left[\vec{E}\cdot\left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi\right) - \vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right) - \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right)\right],$$

where $\vec{E}$, $\vec{B}$, Φ, and $\vec{A}$ are regarded as independent fields. Use their known transformation laws to establish how $\mathcal{L}$ responds to infinitesimal Lorentz transformations.

Problem 3 (30 marks)
The Schwinger-type Lagrange function for a relativistic particle of mass m, in force-free motion, is

$$L = \vec{p}\cdot\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} - \vec{v}\right) + mc\left(c - \sqrt{c^2 - v^2}\right).$$

Show that this gives the familiar nonrelativistic expression for $v \ll c$. Then use the implied relation between $\vec{v}$ and $\vec{p}$ to eliminate the velocity $\vec{v}$ and so find the corresponding Hamilton function $H(\vec{r},\vec{p})$. What is the physical meaning of the action $W_{12} = \int_2^1 \mathrm{d}t\, L$ evaluated for an actual trajectory?

Problem 4 (30 marks)
The charge density of an electric point dipole $\vec{d}$ at rest at $\vec{r} = 0$ is given by

$$\rho(\vec{r}) = -\vec{d}\cdot\vec{\nabla}\delta(\vec{r}).$$

Verify that

$$\int (\mathrm{d}\vec{r})\,\rho(\vec{r}) = 0 \quad \text{and} \quad \int (\mathrm{d}\vec{r})\,\vec{r}\rho(\vec{r}) = \vec{d}.$$

Then find the electrostatic potential $\Phi(\vec{r})$ and the electric field $\vec{E}(\vec{r})$ of the point dipole. You may find it convenient to make use of the dyadic double gradient of $\frac{1}{r}$ that is given by

$$\vec{\nabla}\vec{\nabla}\frac{1}{r} = \frac{3\,\vec{r}\,\vec{r} - r^2\,\overleftrightarrow{1}}{r^5} - \frac{4\pi}{3}\overleftrightarrow{1}\,\delta(\vec{r}).$$

Why do we need to subtract the "contact term" $\frac{4\pi}{3}\overleftrightarrow{1}\,\delta(\vec{r})$?