1

We differentiate and find first

$$\frac{\partial}{\partial t}\vec{G}_{\mathrm{ch}}(\vec{r},t) = \sum_j \delta(\vec{r}-\vec{r}_j(t))\, m_j\dot{\vec{v}}_j(t) - \sum_j \vec{v}_j(t)\cdot\vec{\nabla}\delta(\vec{r}-\vec{r}_j(t))\, m_j\vec{v}_j(t)$$

and

$$\vec{\nabla}\cdot\overset{\leftrightarrow}{T}_{\mathrm{ch}}(\vec{r},t) = \sum_j \vec{\nabla}\delta(\vec{r}-\vec{r}_j(t))\cdot m_j\vec{v}_j(t)\vec{v}_j(t)$$

and then

$$\begin{aligned}\frac{\partial}{\partial t}\vec{G}_{\mathrm{ch}} + \vec{\nabla}\cdot\overset{\leftrightarrow}{T}_{\mathrm{ch}} &= \sum_j \delta(\vec{r}-\vec{r}_j(t))\, m_j\dot{\vec{v}}_j(t) \\ &= \sum_j \delta(\vec{r}-\vec{r}_j(t))\left[e_j\vec{E}(\vec{r}_j(t),t) + e_j\frac{\vec{v}_j(t)}{c}\times\vec{B}(\vec{r}_j(t),t)\right] \\ &= \rho(\vec{r},t)\vec{E}(\vec{r},t) + \frac{1}{c}\vec{j}(\vec{r},t)\times\vec{B}(\vec{r},t) = \vec{f}(\vec{r},t)\,.\end{aligned}$$

Combined with $\frac{\partial}{\partial t}\vec{G} + \vec{\nabla}\cdot\overset{\leftrightarrow}{T} + \vec{f} = 0$, the statement about the momentum density $\vec{G}$ of the electromagnetic field and its current density $\overset{\leftrightarrow}{T}$, we get

$$\frac{\partial}{\partial t}\vec{G}_{\mathrm{tot}} + \vec{\nabla}\cdot\overset{\leftrightarrow}{T}_{\mathrm{tot}} = 0\,,$$

which expresses the local momentum conservation for the total momentum density $\vec{G}_{\mathrm{tot}} = \vec{G}+\vec{G}_{\mathrm{ch}}$ and the total momentum current density $\overset{\leftrightarrow}{T}_{\mathrm{tot}} = \overset{\leftrightarrow}{T}+\overset{\leftrightarrow}{T}_{\mathrm{ch}}$.

2(i)

For an infinitesimal Lorentz transformation we have

$$\begin{aligned}\delta\rho(\vec{r},t) &= -\delta_{\mathrm{coor}}\rho(\vec{r},t) + \frac{\delta\vec{v}}{c^2}\cdot\vec{j}(\vec{r},t) \\ &= -\left(\delta\vec{v}t\cdot\vec{\nabla} + \frac{\delta\vec{v}\cdot\vec{r}}{c^2}\frac{\partial}{\partial t}\right)\rho(\vec{r},t) + \frac{\delta\vec{v}}{c^2}\cdot\vec{j}(\vec{r},t) \\ &= -\delta\vec{v}t\cdot\vec{\nabla}\rho(\vec{r},t) + \frac{1}{c^2}\left[\delta\vec{v}\cdot\vec{r}\,\vec{\nabla}\cdot\vec{j}(\vec{r},t) + \delta\vec{v}\cdot\vec{j}(\vec{r},t)\right] \\ &= \vec{\nabla}\cdot\left[-\delta\vec{v}t\,\rho(\vec{r},t) + \frac{1}{c^2}\vec{j}(\vec{r},t)\,\delta\vec{v}\cdot\vec{r}\right],\end{aligned}$$

that is: $\delta\rho$ is the divergence of a vector field. It follows that

$$\int(\mathrm{d}\vec{r})\,\delta\rho(\vec{r},t) = 0\,,$$

indeed.

2(ii)

For a finite Lorentz transformation with $\vec{v} = v\vec{e}_z$ we have

$$\begin{aligned}\rho'(\vec{r}\,',t') &= \rho'(x',y',z',t') \\ &= \gamma\rho(x',y',\gamma z' - \gamma v t', \gamma t' - \gamma v z'/c^2) \\ &\quad + \gamma(v/c^2) j_z(x',y',\gamma z' - \gamma v t', \gamma t' - \gamma v z'/c^2)\,,\end{aligned}$$

so that

$$\begin{aligned}\int(\mathrm{d}\vec{r}\,')\,\rho'(\vec{r}\,',t') &= \int(\mathrm{d}\vec{r}\,')\,\rho(\vec{r}\,',t'/\gamma - vz'/c^2) \\ &\quad + \frac{v}{c^2}\int(\mathrm{d}\vec{r}\,')\,j_z(\vec{r}\,',t'/\gamma - vz'/c^2)\end{aligned}$$

after substituting $z' \to z'/\gamma + vt'$, $\mathrm{d}z' \to \mathrm{d}z'/\gamma$ in the z' integration.

Now, making use of the hint, we have

$$\begin{aligned}\rho(\vec{r}\,',t'/\gamma - vz'/c^2) &= \rho(\vec{r}\,',t'/\gamma) \\ &\quad - \int \mathrm{d}t\,[\eta(t'/\gamma - t) - \eta(t'/\gamma - vz'/c^2 - t)]\frac{\partial}{\partial t}\rho(\vec{r}\,',t) \\ &= \rho(\vec{r}\,',t'/\gamma) \\ &\quad - \int \mathrm{d}t\,[\eta(t'/\gamma - t) - \eta(t'/\gamma - vz'/c^2 - t)]\vec{\nabla}'\cdot\vec{j}(\vec{r}\,',t) \\ &= \rho(\vec{r}\,',t'/\gamma) + \int \mathrm{d}t\,\vec{j}(\vec{r}\,',t)\cdot\vec{\nabla}'\eta(t'/\gamma - vz'/c^2 - t) \\ &\quad - \vec{\nabla}'\cdot\int \mathrm{d}t\,[\eta(t'/\gamma - t) - \eta(t'/\gamma - vz'/c^2 - t)]\vec{j}(\vec{r}\,',t)\,.\end{aligned}$$

The total divergence has a vanishing integral over all of $\vec{r}\,'$ space, and since $\vec{\nabla}'\eta(t'/\gamma - vz'/c^2 - t) = -(v/c^2)\,\delta(t'/\gamma - vz'/c^2 - t)\,\vec{e}_z$ we get

$$\int(\mathrm{d}\vec{r}\,')\,\rho'(\vec{r}\,',t') = \int(\mathrm{d}\vec{r}\,')\,\rho(\vec{r}\,',t'/\gamma) = \int(\mathrm{d}\vec{r})\,\rho(\vec{r},t)\,,$$

where the last step remembers that the total charge $\int(\mathrm{d}\vec{r})\,\rho(\vec{r},t)$ does not depend on time t.

3

Clearly,

$$\int(\mathrm{d}\vec{r})\,\rho(\vec{r},t) = 0$$

because $\rho(\vec{r},t)$ is a total divergence, and

$$\int(\mathrm{d}\vec{r})\,\vec{r}\,\rho(\vec{r},t) = \vec{d}(t)$$

after an integration by parts: $-\vec{r}\,\vec{d}\cdot\vec{\nabla}\delta(\vec{r}-\vec{R}) = -\vec{\nabla}\cdot\left[\vec{d}\,\delta(\vec{r}-\vec{R})\,\vec{r}\right] + \vec{d}\,\delta(\vec{r}-\vec{R})$.

We have

$$\begin{aligned}
\vec{\nabla}\cdot\vec{j}(\vec{r},t) &= -\frac{\partial}{\partial t}\rho(\vec{r},t) \\
&= -\vec{d}(t)\cdot\vec{\nabla}\,\dot{\vec{R}}(t)\cdot\vec{\nabla}\delta(\vec{r}-\vec{R}(t)) + \dot{\vec{d}}(t)\cdot\vec{\nabla}\delta(\vec{r}-\vec{R}(t)) \\
&= \vec{\nabla}\cdot\left[\dot{\vec{R}}(t)\rho(\vec{r},t) + \dot{\vec{d}}(t)\delta(\vec{r}-\vec{R}(t))\right],
\end{aligned}$$

so that

$$\vec{j}(\vec{r},t) = \dot{\vec{R}}(t)\rho(\vec{r},t) + \dot{\vec{d}}(t)\delta(\vec{r}-\vec{R}(t))\,.$$

It follows that

$$\vec{\mu}(t) = \frac{1}{2c}\int(\mathrm{d}\vec{r})\,\vec{r}\times\vec{j}(\vec{r},t) = \frac{1}{2c}\Big(\vec{d}(t)\times\dot{\vec{R}}(t) + \vec{R}(t)\times\dot{\vec{d}}(t)\Big)\,,$$

telling us that an electric dipole moment in motion is accompanied by a magnetic dipole moment.