Problem 1 (10 marks)
Consider a collection of charges in nonrelativistic motion, so that

$$\begin{aligned}\vec{G}_{\mathrm{ch}}(\vec{r},t) &= \sum_j \delta(\vec{r}-\vec{r}_j(t))\, m_j\vec{v}_j(t)\,,\\ \overleftrightarrow{T}_{\mathrm{ch}}(\vec{r},t) &= \sum_j \delta(\vec{r}-\vec{r}_j(t))\, m_j\vec{v}_j(t)\vec{v}_j(t)\end{aligned}$$

are their momentum density and momentum current density, respectively. Show that

$$\frac{\partial}{\partial t}\vec{G}_{\mathrm{ch}} + \vec{\nabla}\cdot\overleftrightarrow{T}_{\mathrm{ch}} = \vec{f}\,,$$

where $f(\vec{r},t)$ is the familiar Lorentz force density. How does the local momentum conservation follow from this?

Problem 2 (30 marks)
Observer A uses unprimed coordinates, observer B uses primed coordinates. They move relative to each other with velocity $\vec{v}$. By considering both (i) infinitesimal Lorentz transformations and (ii) a finite Lorentz transformation, show that both observers see the same total charge, that is

$$\int(\mathrm{d}\vec{r})\,\rho(\vec{r},t) = \int(\mathrm{d}\vec{r}\,')\,\rho'(\vec{r}\,',t')\,,$$

where the integrations cover all of space.
Hints: You can choose $\vec{v}$ along the z axis if you like; the identity

$$f(x-a) = f(x) - \int \mathrm{d}x'\,[\eta(x-x') - \eta(x-a-x')]\frac{\mathrm{d}f(x')}{\mathrm{d}x'}$$

could be useful.

Problem 3 (20 marks)
A point dipole $\vec{d}(t)$ moves along the trajectory $\vec{R}(t)$, so that the charge density is given by

$$\rho(\vec{r},t) = -\vec{d}(t)\cdot\vec{\nabla}\delta(\vec{r}-\vec{R}(t))\,.$$

Verify that $\int(\mathrm{d}\vec{r})\,\rho(\vec{r},t) = 0$ and $\int(\mathrm{d}\vec{r})\,\vec{r}\,\rho(\vec{r},t) = \vec{d}(t)$. Then find the corresponding current density $\vec{j}(\vec{r},t)$ and express the magnetic dipole moment $\vec{\mu}(t)$ in terms of $\vec{d}(t)$ and $\vec{R}(t)$.