1 The overall effect of the two successive reflections is the mapping

$$\boldsymbol{r} \to \boldsymbol{r} - 2\boldsymbol{e}_1\, \boldsymbol{e}_1 \cdot \boldsymbol{r} \to \big(\boldsymbol{r} - 2\boldsymbol{e}_1\, \boldsymbol{e}_1 \cdot \boldsymbol{r}\big) - 2\boldsymbol{e}_2\, \boldsymbol{e}_2 \cdot \big(\boldsymbol{r} - 2\boldsymbol{e}_1\, \boldsymbol{e}_1 \cdot \boldsymbol{r}\big)\,,$$

that is: $\boldsymbol{r} \to \boldsymbol{r} - 2\boldsymbol{e}_1\, \boldsymbol{e}_1 \cdot \boldsymbol{r} - 2\boldsymbol{e}_2\, \boldsymbol{e}_2 \cdot \boldsymbol{r} + 4\boldsymbol{e}_2\, \boldsymbol{e}_2 \cdot \boldsymbol{e}_1\, \boldsymbol{e}_1 \cdot \boldsymbol{r}$. In the case of $\boldsymbol{e}_1 = \pm\boldsymbol{e}_2$, the second reflection undoes the first and the overall mapping is the identity.

(a) We need to verify that $\boldsymbol{e}_1 \times \boldsymbol{e}_2$ is mapped onto itself, which is immediate.

(b) We take a vector that is perpendicular to the rotation axis, such as $\boldsymbol{e}_1$. This is mapped onto $-\boldsymbol{e}_1 + 2\boldsymbol{e}_2\, \boldsymbol{e}_2 \cdot \boldsymbol{e}_1$, so that

$$\cos\phi = \boldsymbol{e}_1 \cdot \big(-\boldsymbol{e}_1 + 2\boldsymbol{e}_2\, \boldsymbol{e}_2 \cdot \boldsymbol{e}_1\big) = -1 + 2\big(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2\big)^2 = -1 + 2(\cos\alpha)^2 = \cos(2\alpha)\,.$$

This gives $\phi = 2\alpha$ or $\phi = -2\alpha$.

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2 We use (2.2.140) for $\gamma = 0$, $\omega = \omega_0$, $x_0 = 0$ and $F(t') = mg$,

$$x(t) = v_0 \frac{\sin(\omega_0 t)}{\omega_0} + g \int_0^t \mathrm{d}t'\, \frac{\sin(\omega_0(t-t'))}{\omega_0} = v_0 \frac{\sin(\omega_0 t)}{\omega_0} + \frac{g}{\omega_0^2}\big[1 - \cos(\omega_0 t)\big]\,,$$

where we need to find v_0 such that $\dot{x}(T) = 0$. This requirement reads

$$v_0 \cos(\omega_0 T) + \frac{g}{\omega_0} \sin(\omega_0 T) = 0 \quad \text{or} \quad v_0 = -\frac{g}{\omega_0} \tan(\omega_0 T)\,,$$

so that we have

$$x(t) = \frac{g}{\omega_0^2} - \frac{g}{\omega_0^2} \frac{\sin(\omega_0 t)\sin(\omega_0 T) + \cos(\omega_0 t)\cos(\omega_0 T)}{\cos(\omega_0 T)} = \frac{g}{\omega_0^2} - \frac{g}{\omega_0^2} \frac{\cos\big(\omega_0(T-t)\big)}{\cos(\omega_0 T)}$$

and

$$\dot{x}(t) = -\frac{g}{\omega_0} \frac{\sin\big(\omega_0(T-t)\big)}{\cos(\omega_0 T)}$$

at intermediate times.

3

(a) For $E > 0$ the motion is unbounded; for $0 > E > -E_0$ we have motion between two turning points; there is no energy range with motion bounded by one turning point; $E < -E_0$ is not possible.

(b) For $E = V(\pm a) = -E_0/[\cosh(ka)]^2$, (3.1.19) gives

$$\begin{aligned} T(E) &= 2\int_{-a}^{a} \mathrm{d}x \left[\frac{2E_0}{m}\left(\frac{1}{\cosh(kx)^2} - \frac{1}{\cosh(ka)^2}\right)\right]^{-1/2} \\ &= \sqrt{\frac{2m}{E_0}} \int_{-a}^{a} \mathrm{d}x \, \frac{\cosh(ka)\cosh(kx)}{\sqrt{\cosh(ka)^2 - \cosh(kx)^2}} \\ &= \sqrt{\frac{2m}{E_0}} \frac{\cosh(ka)}{k} \underbrace{\int_{-a}^{a} \mathrm{d}x \, \frac{k\cosh(kx)}{\sqrt{\sinh(ka)^2 - \sinh(kx)^2}}}_{=\pi}, \end{aligned}$$

where the integral can be evaluated with the substitution $\sinh(kx) = \sinh(ka)\sin\varphi$. With $\cosh(ka) = \sqrt{-E_0/E}$, the final answer is

$$T(E) = \frac{\pi}{k}\sqrt{\frac{2m}{-E}} \qquad \text{for } -E_0 < E < 0\,.$$

4

(a) The curl of $\boldsymbol{F}$ is

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r}) &= \underbrace{\boldsymbol{\nabla} f_1(r)}_{=\frac{1}{r}f_1'(r)\boldsymbol{r}} \times \boldsymbol{a} + \underbrace{\boldsymbol{\nabla} f_2(r)}_{=\frac{1}{r}f_2'(r)\boldsymbol{r}} \times (\boldsymbol{a}\cdot\boldsymbol{r}\,\boldsymbol{r}) + f_2(r)\big(\underbrace{\boldsymbol{\nabla}\boldsymbol{a}\cdot\boldsymbol{r}}_{=\boldsymbol{a}}\big) \times \boldsymbol{r} \\ &\quad + f_2(r)\boldsymbol{a}\cdot\boldsymbol{r}\,\underbrace{\boldsymbol{\nabla}\times\boldsymbol{r}}_{=0} = \left(\frac{1}{r}f_1'(r) - f_2(r)\right)\boldsymbol{r}\times\boldsymbol{a}\,. \end{aligned}$$

For a conservative force, we need a vanishing curl, which requires $f_2(r) = \frac{1}{r}f_1'(r)$.

(b) Now, for $f_2(r) = \frac{1}{r}f_1'(r)$, we have

$$\boldsymbol{F}(\boldsymbol{r}) = f_1(r)\boldsymbol{a} + \frac{1}{r}f_1'(r)\boldsymbol{a}\cdot\boldsymbol{r}\,\boldsymbol{r} = f_1(r)\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{r}) + \boldsymbol{a}\cdot\boldsymbol{r}\boldsymbol{\nabla} f_1(r) = \boldsymbol{\nabla}\big(f_1(r)\boldsymbol{a}\cdot\boldsymbol{r}\big)\,,$$

so that $V(\boldsymbol{r}) = -f_1(r)\boldsymbol{a}\cdot\boldsymbol{r}$ is a potential energy for this conservative force.
