1 According to Section 4.5 of the lecture notes, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} &= \frac{1}{M}\boldsymbol{P}_{\mathrm{tot}}\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}_{\mathrm{tot}} &= \sum_{j=1}^{J}\boldsymbol{F}_j^{(\mathrm{ext})} = M\boldsymbol{g}\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{tot}} &= \sum_{j=1}^{J}\boldsymbol{v}_j\cdot\boldsymbol{F}_j^{(\mathrm{ext})} = \boldsymbol{P}_{\mathrm{tot}}\cdot\boldsymbol{g}\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_{\mathrm{tot}} &= \sum_{j=1}^{J}\boldsymbol{r}_j\times\boldsymbol{F}_j^{(\mathrm{ext})} = M\boldsymbol{R}\times\boldsymbol{g}\,.
\end{aligned}$$

These are solved by

$$\begin{aligned}
\boldsymbol{R}(t) &= \boldsymbol{R}_0 + \frac{\boldsymbol{P}_0}{M}(t-t_0) + \frac{1}{2}\boldsymbol{g}(t-t_0)^2\,,\\
\boldsymbol{P}_{\mathrm{tot}}(t) &= \boldsymbol{P}_0 + M\boldsymbol{g}(t-t_0)\,,\\
E_{\mathrm{tot}}(t) &= E_0 + \boldsymbol{P}_0\cdot\boldsymbol{g}(t-t_0) + \frac{1}{2}Mg^2(t-t_0)^2\,,\\
\boldsymbol{L}_{\mathrm{tot}}(t) &= \boldsymbol{L}_0 + M\boldsymbol{R}_0\times\boldsymbol{g}(t-t_0) + \frac{1}{2}\boldsymbol{P}_0\times\boldsymbol{g}(t-t_0)^2\,.
\end{aligned}$$

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2 Scattering occurs only when $b < R$. For $b < R$, then, we denote the angle of incidence by α. It is related to the impact parameter by $b = R\sin\alpha$ and to the scattering angle by $\theta = \pi - 2\alpha$. Therefore,

$$\cos\theta = -\cos(2\alpha) = 2(\sin\alpha)^2 - 1 = 2(b/R)^2 - 1\,,$$

which implies

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2}\left|\frac{\mathrm{d}b^2}{\mathrm{d}\cos\theta}\right| = \frac{1}{4}R^2$$

for the differential cross section and $\sigma = \pi R^2$ for the total cross section.

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3 According to Section 7.3 of the lecture notes, we have

$$x = R(\phi - \sin\phi)\,,$$
$$y = R(1 - \cos\phi)$$

for the brachistochrone, here with $0 \le \phi \le 2\pi$ and $R = a/(2\pi)$. With

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = R\mathrm{d}\phi\,\sqrt{2 - 2\cos\phi} = 2R\mathrm{d}\phi\,\sin\frac{\phi}{2}$$

and

$$v = \sqrt{2gy} = 2\sqrt{gR}\sin\frac{\phi}{2}$$

we get

$$T = \int \frac{\mathrm{d}s}{v} = \int_0^{2\pi} \mathrm{d}\phi\,\frac{R}{\sqrt{gR}} = 2\pi\sqrt{R/g}$$

for the duration and

$$S = \int \mathrm{d}s = 2R\int_0^{2\pi} \mathrm{d}\phi\,\sin\frac{\phi}{2} = 8R$$

for the distance covered. Their ratio is the average speed,

$$\text{average speed} = \frac{S}{T} = \frac{4}{\pi}\sqrt{gR} = \frac{4}{\pi}\sqrt{\frac{ga}{2\pi}}\,.$$

4

(a) We have the Lagrange function

$$L(t, x, \dot{x}) = \frac{m}{2}\dot{x}^2 - \frac{k}{2}\Big(\sqrt{x^2 + a^2} - a\Big)^2\,;$$

it has no parametric t dependence. The energy

$$E = \frac{m}{2}\dot{x}^2 + \frac{k}{2}\Big(\sqrt{x^2 + a^2} - a\Big)^2$$

is conserved. The equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}m\dot{x} = -k\Big(\sqrt{x^2 + a^2} - a\Big)\frac{x}{\sqrt{x^2 + a^2}} = -kx + \frac{kax}{\sqrt{x^2 + a^2}}\,.$$

(b) Since $\dot{x} = a\dot{\vartheta}\cosh\vartheta$ and $\sqrt{x^2+a^2} = a\cosh\vartheta$, we have the Lagrange function

$$L(t,\vartheta,\dot{\vartheta}) = \frac{ma^2}{2}\big(\dot{\vartheta}\cosh\vartheta\big)^2 - \frac{ka^2}{2}(\cosh\vartheta - 1)^2\,.$$

This gives the equation of motion

$$m\frac{\mathrm{d}}{\mathrm{d}t}\big[\dot{\vartheta}(\cosh\vartheta)^2\big] = m\dot{\vartheta}^2\cosh\vartheta\sinh\vartheta - k(\cosh\vartheta-1)\sinh\vartheta$$

or

$$\ddot{\vartheta}(\cosh\vartheta)^2 + \dot{\vartheta}^2\cosh\vartheta\sinh\vartheta = -\frac{k}{m}(\cosh\vartheta-1)\sinh\vartheta\,.$$

(c) For $|x| \ll a$ we have $\sqrt{x^2+a^2} - a = \dfrac{x^2}{2a} + \cdots$ and get the approximate Lagrange function

$$L(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - \frac{kx^4}{8a^2}$$

and the equation of motion

$$m\ddot{x} = -\frac{kx^3}{2a^2}\,.$$

Since $x = a\vartheta$ here, we also have $L(\vartheta,\dot{\vartheta}) = \dfrac{ma^2}{2}\dot{\vartheta}^2 - \dfrac{ka^2\vartheta^4}{8}$ and $\ddot{\vartheta} = -\dfrac{k}{2m}\vartheta^3$.
