1

(a) From $m\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\gamma t}\dot{x}\right) = \mathrm{e}^{\gamma t}F(t)$ we get first

$$\dot{x}(t'') = \int_0^{t''} \mathrm{d}t' \; \mathrm{e}^{-\gamma(t'' - t')} \frac{1}{m} F(t')$$

and then

$$x(t) = \int_0^t \mathrm{d}t'' \, \dot{x}(t'') = \int_0^t \mathrm{d}t' \, \underbrace{\gamma \int_{t'}^t \mathrm{d}t'' \; \mathrm{e}^{-\gamma(t'' - t')}}_{= 1 - \mathrm{e}^{-\gamma(t - t')}} \frac{1}{\gamma m} F(t')$$

$$= \int_0^t \mathrm{d}t' \, \frac{1 - \mathrm{e}^{-\gamma(t - t')}}{\gamma} \frac{F(t')}{m} \, .$$

Alternatively, one could just use the $\omega_0 \to 0$ limit of the Green's function for the damped harmonic oscillator.

(b) Either by using the expression of (a) or the ansatz $x(t) = A\cos(\omega t) + B\sin(\omega t)$, one finds

$$x(t) = \frac{a/\omega}{\omega^2 + \gamma^2}\left[\gamma \sin(\omega t) - \omega \cos(\omega t)\right] .$$

2

(a) The potential energy has its minimum at $x = 0$, and for $|x| \ll a$ we have

$$V(x) \cong V_0 \frac{x^2}{a^2} = \frac{1}{2} m \omega_0^2 x^2 \, ,$$

so that $\omega_0^2 = 2V_0/(ma^2)$ and the period is $T = 2\pi/\omega_0 = \pi a \sqrt{2m/V_0}$.

(b) All positive energies are permissible. We write $E = V_0[\tan(x_0/a)]^2$ and use the hint to arrive at

$$T(E) = 2\int_{-x_0}^{x_0} \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}\big[E - V(x)\big]}}$$
$$= \sqrt{\frac{2m}{V_0}}\, a\cos(x_0/a) \underbrace{\int_{-x_0}^{x_0} \frac{\mathrm{d}x}{a}\frac{\cos(x/a)}{\sqrt{\big[\sin(x_0/a)\big]^2 - \big[\sin(x/a)\big]^2}}}_{=\pi}$$
$$= \pi a\sqrt{\frac{2m}{V_0 + E}}$$

with $\cos(x_0/a) = \sqrt{V_0/(V_0+E)}$ in the last step.

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3 Except for an overall factor of -3, the main difference between

$$\mathbf{I} = \int (\mathrm{d}\boldsymbol{r})\,\rho(\boldsymbol{r})\big(r^2\mathbf{1} - \boldsymbol{r}\,\boldsymbol{r}\big) \quad \text{and} \quad \mathbf{Q} = \int (\mathrm{d}\boldsymbol{r})\,\rho(\boldsymbol{r})\big(3\boldsymbol{r}\,\boldsymbol{r} - r^2\mathbf{1}\big)$$

is that $\mathbf{Q}$ is traceless, whereas $\operatorname{tr}\{\mathbf{I}\} = \int (\mathrm{d}\boldsymbol{r})\,\rho(\boldsymbol{r})\,2r^2 > 0$. It follows that

$$\mathbf{Q} = \operatorname{tr}\{\mathbf{I}\}\,\mathbf{1} - 3\mathbf{I}\,.$$

4

(a) We have

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x,y) \quad \text{and} \quad H = \frac{1}{2m}(p_x^2 + p_y^2) + V(x,y)$$

with the potential energy

$$V(x,y) = \frac{k}{2}\Big(\sqrt{(x+a)^2 + y^2} - a\Big)^2 + \frac{k}{2}\Big(\sqrt{x^2 + (y+a)^2} - a\Big)^2$$
$$+ \frac{k}{2}\Big(\sqrt{(x - a\cos\theta_0)^2 + (y - a\sin\theta_0)^2} - a\Big)^2.$$

(b) Clearly, $V(x,y) \geq 0$ and $V(x,y) = 0$ only for $(x,y) = (0,0)$. All springs have their natural length when the point mass is at $(x,y) = (0,0)$.

(c) For $|x| \ll a$ and $|y| \ll a$, we have

$$V(x,y) \cong \frac{k}{2}\left[x^2 + y^2 + (x\cos\theta_0 + y\sin\theta_0)^2\right].$$

The characteristic frequencies ω_1 and ω_2 are, therefore, such that the determinant of the 2×2 matrix in

$$\begin{pmatrix} \omega^2 - \omega_0^2[1 + (\cos\theta_0)^2] & -\omega_0^2 \sin\theta_0 \cos\theta_0 \\ -\omega_0^2 \sin\theta_0 \cos\theta_0 & \omega^2 - \omega_0^2[1 + (\sin\theta_0)^2] \end{pmatrix} X = 0$$

vanishes for $\omega = \omega_1$ and $\omega = \omega_2$. This requires $\omega_1^2 + \omega_2^2 = 3\omega_0^2$ and $\omega_1^2\omega_2^2 = 2\omega_0^4$, so that $\omega_1 = \omega_0$ and $\omega_2 = \sqrt{2}\,\omega_0$. The respective mode amplitudes are

$$X_1 = \begin{pmatrix} -\sin\theta_0 \\ \cos\theta_0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix}.$$

(d) The slower normal mode "1" is an oscillation perpendicular to the line-of-sight from $(0,0)$ to $(x_3, y_3) = (a\cos\theta_0, a\sin\theta_0)$, and the faster normal mode "2" is an oscillation along the line-of-sight from $(0,0)$ to (x_3, y_3).