1. Time-dependent force (25=15+10 marks)
A point mass m is moving along the x axis under the influence of a time-dependent force $F(t)$ and a frictional force $-m\gamma\dot{x}$.

(a) Solve the equation of motion

$$m\ddot{x} = -m\gamma\dot{x} + F(t)$$

to find $x(t)$ for the initial conditions $x(t=0) = 0$, $\dot{x}(t=0) = 0$.

(b) For the harmonic force $F(t) = ma\cos(\omega t)$ with constant acceleration a and constant angular frequency ω, what is $x(t)$ for late times?

Hint: What is the time derivative of $\mathrm{e}^{\gamma t}\dot{x}(t)$?

2. One-dimensional periodic motion (25=10+15 marks)
For $-\frac{1}{2}\pi a < x < \frac{1}{2}\pi a$, the potential energy for a point mass m is

$$V(x) = V_0\big[\tan(x/a)\big]^2 ,$$

where a and V_0 are positive constants.

(a) What is the period of small-amplitude oscillations?

(b) Find the energy-dependent period $T(E)$ for all permissible values of the energy E.

Hint: The identity $(\tan\alpha)^2 - (\tan\beta)^2 = \dfrac{(\sin\alpha)^2 - (\sin\beta)^2}{(\cos\alpha\ \cos\beta)^2}$ could be useful.

3. Mass distribution (20 marks)
A body has mass density $\rho(\boldsymbol{r})$ with the center-of-mass at $\boldsymbol{r} = 0$. How are the inertia dyadic $\mathbf{I}$ and the quadrupole dyadic $\mathbf{Q}$ related to each other?

4. Normal modes (30=8+6+10+6 marks)
A point mass m is moving without friction in the horizontal x, y plane. Three equal springs (spring constant $k = m\omega_0^2$, natural length a) are used to attach the point mass to $(x_1, y_1) = (-a, 0)$, $(x_2, y_2) = (0, -a)$, and $(x_3, y_3) = (a\cos\theta_0, a\sin\theta_0)$ with $0 < \theta_0 < \frac{1}{2}\pi$. The masses of the springs are negligibly small.

(a) What is the Lagrange function for this situation?

(b) Verify that the potential energy has its minimum at $(x, y) = (0, 0)$.

(c) Find the normal modes and their characteristic frequencies.

(d) Use words and suitable sketches to describe the normal modes.