$\boxed{1}$

(a) The potential energy $ma|x|$ gives rise to the force $F = -ma\dfrac{\partial}{\partial x}|x| = -ma\,\mathrm{sgn}(x)$; it follows that the stated energy is the correct conserved energy for the given force.

(b) We have motion with constant acceleration $-a$ for $x > 0$ and constant acceleration a for $x < 0$. Let's take $x = 0$ and $\dot{x} = v_0 > 0$ at $t = 0$, then $E = \frac{1}{2}mv_0^2$ and $\dot{x} = v_0 - at$ for the half-period $0 < t < \frac{1}{2}T$ of the motion, and either $\dot{x}(t = T/4) = 0$ or $\dot{x}(t = T/2) = -v_0$ tell us that $aT = 4v_0$. Accordingly, the period is $T(E) = \dfrac{4}{a}\sqrt{2E/m}$. — This answer is also available as the $\nu = 1$ case of Exercise 24 with $\kappa = ma$.

(c) Since $\overline{E_{\mathrm{kin}}} + \overline{E_{\mathrm{pot}}} = E$, it is enough to calculate $\overline{E_{\mathrm{kin}}}$, and averaging over the quarter-period $0 < t < T/4$ is as good as averaging over the full period. Thus,

$$\overline{E_{\mathrm{kin}}} = \frac{4}{T}\int_0^{T/4} \mathrm{d}t\,\frac{m}{2}(v_0 - at)^2 = \frac{4}{T}\frac{m}{2}\frac{v_0^3}{3a} = \frac{2}{3}\frac{mv_0^2}{aT/v_0} = \frac{2}{3}\frac{2E}{4} = \frac{1}{3}E$$

and $\overline{E_{\mathrm{pot}}} = \dfrac{2}{3}E$.

$\boxed{2}$ The solution to the equation of motion is given in (2.2.19) on page 41 of the lecture notes, that is

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_\infty t + (\boldsymbol{v}_0 - \boldsymbol{v}_\infty)\frac{1 - \mathrm{e}^{-\gamma t}}{\gamma}$$

with $\boldsymbol{v}_\infty = \boldsymbol{g}/\gamma$, and $\boldsymbol{r}(T) = 0$ establishes

$$\boldsymbol{r}_0 = -\boldsymbol{v}_\infty T - (\boldsymbol{v}_0 - \boldsymbol{v}_\infty)\frac{1 - \mathrm{e}^{-\gamma T}}{\gamma}\,,$$

so that

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{v}_\infty(t - T) + (\boldsymbol{v}_0 - \boldsymbol{v}_\infty)\frac{\mathrm{e}^{-\gamma T} - \mathrm{e}^{-\gamma t}}{\gamma} \\ &= \boldsymbol{g}\frac{t - T}{\gamma} + \left(\boldsymbol{v}_0 - \frac{1}{\gamma}\boldsymbol{g}\right)\frac{\mathrm{e}^{-\gamma T} - \mathrm{e}^{-\gamma t}}{\gamma}\,.\end{aligned}$$

An alternative solution could begin with the result of Exercise 18 and apply it to the current situation.

$\boxed{3}$ The force

$$F(x) = -\frac{\partial}{\partial x}V(x) = E_0 a^2 \frac{2x(x-2a)(x+2a)}{\left(x^2+2a^2\right)^3}$$

vanishes at $x=0$, $x=2a$, and $x=-2a$. At these positions, the potential energy has the values

$$V(0) = E_0 a^2 \frac{-a^2}{(2a^2)^2} = --\frac{1}{4}E_0\,, \quad V(\pm 2a) = E_0 a^2 \frac{(2a)^2 - a^2}{[(2a)^2+2a^2]^2} = \frac{1}{12}E_0\,,$$

and we note that $V(x \to \pm\infty) = 0$. Near the points of vanishing force, the force is approximated by

$$\begin{aligned}
x \simeq 0 \,:\; F(x) &\simeq E_0 a^2 \frac{2x(-2a)(+2a)}{\left(2a^2\right)^3} = -\frac{E_0}{a^2}x\,,\\
x \simeq 2a \,:\; F(x) &\simeq E_0 a^2 \frac{4a(x-2a)(2a+2a)}{\left((2a)^2+2a^2\right)^3} = \frac{2}{27}\frac{E_0}{a^2}(x-2a)\,,\\
x \simeq -2a \,:\; F(x) &\simeq E_0 a^2 \frac{-4a(-2a-2a)(x+2a)}{\left((-2a)^2+2a^2\right)^3} = \frac{2}{27}\frac{E_0}{a^2}(x+2a)\,,
\end{aligned}$$

so that

$$F'(x) = -\frac{\partial^2}{\partial x^2}V(x) = \begin{cases} -\dfrac{E_0}{a^2} & \text{for} \quad x = 0\,,\\[2ex] \dfrac{2}{27}\dfrac{E_0}{a^2} & \text{for} \quad x = \pm 2a\,. \end{cases}$$

(a) We have the potential minimum at $x=0$ and maxima at $x=\pm 2a$. There is no motion with $E < -\frac{1}{4}E_0$, and we have

$$\begin{aligned}
-\frac{1}{4}E_0 < E < 0 \,&:\; \text{2 turning points,}\\
0 < E < \frac{1}{12}E_0 \,&:\; \text{2 turning points or 1 turning point,}\\
\frac{1}{12}E_0 < E \,&:\; \text{no turning points.}
\end{aligned}$$

(b) Near $x=0$, the force is $F(x) = -m\omega^2 x$ with $\omega^2 = E_0/(ma^2)$, so that we have the period

$$\frac{2\pi}{\omega} = \frac{2\pi a}{\sqrt{E_0/m}}\,.$$

(c) For $E_0 = -|E_0| < 0$, we have the potential maximum at $x = 0$ and two symmetric minima at $x = \pm 2a$. Now, there is no motion with $E < -\frac{1}{12}|E_0|$ and we have

$$\begin{aligned} -\frac{1}{12}|E_0| < E < 0 &: \text{ 2 turning points,} \\ 0 < E < \frac{1}{4}|E_0| &: \text{ 1 turning point,} \\ \frac{1}{4}|E_0| < E &: \text{ no turning points.} \end{aligned}$$

The force near the minimum at $x = \pm 2a$ is $F(x) = -m\omega^2(x \mp 2a)$ with $\omega^2 = \frac{2}{27}\frac{|E_0|}{ma^2}$, so that we have the period

$$\frac{2\pi}{\omega} = \frac{2\pi a}{\sqrt{\frac{2}{27}|E_0|/m}} \,.$$

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(a) We calculate the respective curls.

(i) This force is conservative:

$$\boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r}) \mathrel{\widehat{=}} \begin{pmatrix} \frac{\partial}{\partial y}(ky + 2kz) - \frac{\partial}{\partial z}(kx + kz) \\ \frac{\partial}{\partial z}(2kx + ky) - \frac{\partial}{\partial x}(ky + 2kz) \\ \frac{\partial}{\partial x}(kx + kz) - \frac{\partial}{\partial y}(2kx + ky) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ;$$

(ii) This for is not conservative:

$$\boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{b} \times \boldsymbol{\nabla} r = -\boldsymbol{b} \times \frac{\boldsymbol{r}}{r} \neq 0\,;$$

(iii) This force is conservative:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r}) &= \boldsymbol{\nabla} \times \frac{\boldsymbol{a}}{r} - \boldsymbol{\nabla} \times \frac{\boldsymbol{r}\,\boldsymbol{r}\cdot\boldsymbol{a}}{r^3} \\ &= -\boldsymbol{a} \times \boldsymbol{\nabla}\frac{1}{r} - \frac{\boldsymbol{r}\cdot\boldsymbol{a}}{r^3}\underbrace{\boldsymbol{\nabla}\times\boldsymbol{r}}_{=0} + \boldsymbol{r} \times \boldsymbol{\nabla}\frac{\boldsymbol{r}\cdot\boldsymbol{a}}{r^3} \\ &= \boldsymbol{a} \times \frac{\boldsymbol{r}}{r^3} + \boldsymbol{r} \times \left(\frac{\boldsymbol{\nabla}\boldsymbol{r}\cdot\boldsymbol{a}}{r^3} + \boldsymbol{r}\cdot\boldsymbol{a}\boldsymbol{\nabla}\frac{1}{r^3}\right) \\ &= \boldsymbol{a} \times \frac{\boldsymbol{r}}{r^3} + \boldsymbol{r} \times \left(\frac{\boldsymbol{a}}{r^3} - \boldsymbol{r}\cdot\boldsymbol{a}\frac{3\boldsymbol{r}}{r^5}\right) = 0\,. \end{aligned}$$

(b) There is no potential energy for force (ii). The other cases are

$$\text{(i)}\quad V(\boldsymbol{r}) = -kx^2 - kxy - kyz - kz^2\,;$$
$$\text{(iii)}\quad V(\boldsymbol{r}) = -\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r}\,.$$
