Problem 1 (20=5+5+10 marks)
A point mass m is moving along the x axis under the influence of the force

$$F = -ma\,\mathrm{sgn}(x) = \begin{cases} -ma & \text{for} \quad x > 0 \\ ma & \text{for} \quad x < 0 \end{cases}$$

with constant $a > 0$.

(a) Show that the energy $E = \dfrac{m}{2}v^2 + ma|x|$ is constant in time.

(b) What is the energy-dependent period $T(E)$ of the periodic motion?

(c) The period-averages of the kinetic energy and the potential energy are

$$\overline{E_{\mathrm{kin}}} = \frac{1}{T}\int_0^T \mathrm{d}t\,\frac{m}{2}v^2 \quad \text{and} \quad \overline{E_{\mathrm{pot}}} = \frac{1}{T}\int_0^T \mathrm{d}t\,ma|x|\,.$$

Express each of them in terms of energy E.

Problem 2 (15 marks)
The constant force $m\boldsymbol{g}$ and the frictional force $-m\gamma\boldsymbol{v}$ are acting on point mass m. The point mass has velocity $\boldsymbol{v}_0$ at time $t = 0$ and is at position $\boldsymbol{r} = 0$ at time T. What is $\boldsymbol{r}(t)$ for $0 < t < T$?

Problem 3 (40=15+10+15 marks)
A point mass m is moving along the x axis under the influence of the force associated with the potential energy

$$V(x) = E_0 a^2 \frac{x^2 - a^2}{(x^2 + 2a^2)^2} \quad \text{with} \quad E_0 > 0 \quad \text{and} \quad a > 0.$$

(a) For which energy ranges do you have motion with one, two, or no turning points?

(b) For the oscillatory motion between two turning points, what is the period of small-amplitude oscillations?

(c) Answer the same questions for $E_0 < 0$.

Problem 4 (25=15+10 marks)

(a)   Which of the following three force fields is conservative?

$$(\text{i}) \quad \boldsymbol{F} \,\widehat{=}\, \begin{pmatrix} 2kx + ky \\ kx + kz \\ ky + 2kz \end{pmatrix} \quad \text{with} \quad k = \text{constant};$$

$$(\text{ii}) \quad \boldsymbol{F} = r\boldsymbol{b} \quad \text{with} \quad \boldsymbol{b} = \text{constant};$$

$$(\text{iii}) \quad \boldsymbol{F} = \frac{r^2 \boldsymbol{a} - \boldsymbol{r}\, \boldsymbol{r} \cdot \boldsymbol{a}}{r^3} \quad \text{with} \quad \boldsymbol{a} = \text{constant}.$$

(b)   Find a potential energy for the conservative forces.