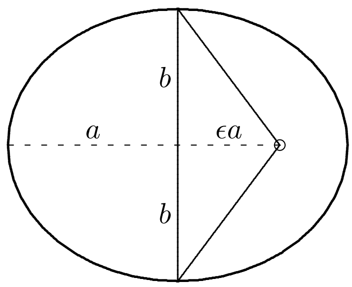
$\boxed{1}$



According to Kepler's Second Law, the time that the planet needs to cover the aphelion-side half of the ellipse is proportional to the area composed of the left half of the ellipse and the triangle. The time spent on the perihelion-side half of the ellipse is proportional to the area of the right half of the ellipse with the triangle removed.

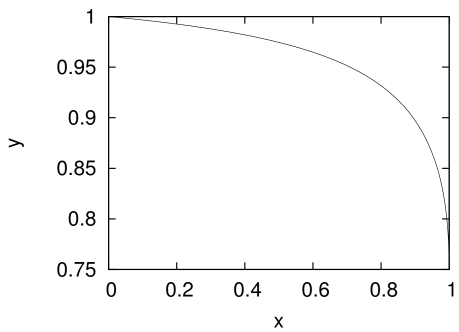
The ellipse has area πab, the triangle has area ϵab. Therefore, it takes the fraction $\dfrac{\frac{1}{2}\pi ab - \epsilon ab}{\pi ab} = \dfrac{1}{2} - \dfrac{\epsilon}{\pi}$ of the round-trip time to cover the perihelion-side half of the ellipse, and it takes the fraction $\dfrac{1}{2} + \dfrac{\epsilon}{\pi}$ to cover the aphelion-side half.

$\boxed{2}$

(a) When entering the ray is deflected by angle $\alpha - \beta$, and again by the same amount when exiting. Therefore, we have $\frac{1}{2}\theta = \alpha - \beta$ with $\sin\alpha = b/R = \sqrt{x}$ and $\sin\beta = \dfrac{3}{4}\sqrt{x}$, so that

$$y = \cos\left(\frac{1}{2}\theta\right) = \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta = \sqrt{1-x}\sqrt{1-\frac{9x}{16}} + \frac{3x}{4}\,.$$

We have $y \cong 1 - \dfrac{x}{32}$ for $0 \lesssim x \ll 1$ and $y = \dfrac{3}{4}$ for $x = 1$; in view of the $\sqrt{1-x}$ factor, the slope $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is infinite at $x = 1$. Here is the graph of $y(x)$:



(b) With $b^2 = R^2 x$ and $\cos\theta = 2\cos\left(\frac{1}{2}\theta\right)^2 - 1 = 2y^2 - 1$ we have $\mathrm{d}b^2 = R^2\mathrm{d}x$ and $\mathrm{d}\cos\theta = 4y\mathrm{d}y$, so that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2}\left|\frac{\mathrm{d}b^2}{\mathrm{d}\cos\theta}\right| = \frac{R^2}{8y}\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = -\frac{R^2}{8y}\frac{\mathrm{d}x}{\mathrm{d}y},$$

where we recognize that $\frac{\mathrm{d}x}{\mathrm{d}y} < 0$. We express x as a function of y,

$$x = 16\frac{1-y^2}{25-24y},$$

and differentiate to arrive at

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4(4y-3)(4-3y)}{y(25-24y)^2}R^2 \quad \text{with} \quad \frac{3}{4} \le y = \cos\frac{\theta}{2} \le 1\,.$$

(c) With $\mathrm{d}\Omega = \mathrm{d}\phi\,\sin\theta\,\mathrm{d}\theta = -\mathrm{d}\phi\,\mathrm{d}\cos\theta = -\mathrm{d}\phi\,4y\,\mathrm{d}y$, we have

$$\begin{aligned}\sigma &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\,\sin\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \int_0^{2\pi}\mathrm{d}\phi\int_{3/4}^{1}\mathrm{d}y\,4y\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\\ &= \int_0^{2\pi}\mathrm{d}\phi\int_{3/4}^{1}\mathrm{d}y\,4y\,\frac{R^2}{8y}\left(-\frac{\mathrm{d}x}{\mathrm{d}y}\right) = 2\pi\frac{R^2}{2}(-x)\Big|_{y=3/4}^{1} = \pi R^2\,.\end{aligned}$$

As expected, the total cross section is the cross-sectional area of the water drop.

$\boxed{3}$

(a) We choose the coordinate system such that $\boldsymbol{r} = 0$ is the position of the center-of-mass, so that $\boldsymbol{r}_1 = -\frac{1}{2}\boldsymbol{a}$ and $\boldsymbol{r}_2 = \frac{1}{2}\boldsymbol{a}$. According to Newton's Shell Theorem, we then have the gravitational potential

$$-\frac{\frac{1}{2}GM}{\left|\boldsymbol{r}+\frac{1}{2}\boldsymbol{a}\right|} - \frac{\frac{1}{2}GM}{\left|\boldsymbol{r}-\frac{1}{2}\boldsymbol{a}\right|} = -G\int(\mathrm{d}\boldsymbol{r}')\frac{\frac{1}{2}M\delta(\boldsymbol{r}'+\frac{1}{2}\boldsymbol{a}) + \frac{1}{2}M\delta(\boldsymbol{r}'-\frac{1}{2}\boldsymbol{a})}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

for points $\boldsymbol{r}$ outside the two balls. It is *as if* we had two point masses $\frac{1}{2}M$ at $\pm\frac{1}{2}\boldsymbol{a}$, with the as-if mass density

$$\rho(\boldsymbol{r}') = \frac{1}{2}M\delta(\boldsymbol{r}'+\tfrac{1}{2}\boldsymbol{a}) + \frac{1}{2}M\delta(\boldsymbol{r}'-\tfrac{1}{2}\boldsymbol{a})\,.$$

The resulting quadrupole moment dyadic is

$$\begin{aligned}\mathsf{Q} &= \int(\mathrm{d}\boldsymbol{r}')\rho(\boldsymbol{r}')\Big(3\boldsymbol{r}'\,\boldsymbol{r}' - r'^2\mathbf{1}\Big)\\ &= 2\times\frac{1}{2}M\Big(3(\tfrac{1}{2}\boldsymbol{a})(\tfrac{1}{2}\boldsymbol{a}) - (\tfrac{1}{2}\boldsymbol{a})^2\mathbf{1}\Big) = \frac{1}{4}M\Big(3\boldsymbol{a}\,\boldsymbol{a} - a^2\mathbf{1}\Big)\,.\end{aligned}$$

(b) At time $t = 0$, each ball is at distance $s(0) = \frac{1}{2}a$ from the center-of-mass that is half-way between the balls. At time $t = T$, the balls touch so that each ball is at distance $s(T) = R$ from the center-of-mass. Each ball is accelerated by the force $G(\frac{1}{2}M)^2/(2s)^2$ toward the center-of-mass, so that

$$\frac{1}{2}M\ddot{s} = -\frac{GM^2}{16s^2} \quad \text{or} \quad \ddot{s} = \frac{\partial}{\partial s}\frac{GM}{8s}\,.$$

It follows that

$$\dot{s}^2 - \frac{GM}{4s} = -\frac{GM}{2a} = \text{constant},$$

with the value of this constant determined by $s(0) = \frac{1}{2}a$ and $\dot{s}(0) = 0$. Since $\dot{s}(t) < 0$ for $t > 0$, we have

$$\dot{s} = \frac{\mathrm{d}s}{\mathrm{d}t} = -\sqrt{\frac{GM}{2a}}\sqrt{\frac{a/2 - s}{s}}$$

and

$$T = \int_0^T \mathrm{d}t = \sqrt{\frac{2a}{GM}}\int_R^{a/2} \mathrm{d}s\,\sqrt{\frac{s}{a/2 - s}} = \sqrt{\frac{a^3/2}{GM}}\int_{2R/a}^{1} \mathrm{d}x\,\sqrt{\frac{x}{1 - x}}\,.$$

For $a \gg R$, the x integral gives $\frac{1}{2}\pi$, and $T \simeq \pi\sqrt{\frac{(a/2)^3}{GM}}$ follows.

$\boxed{4}$ Along the path specified by $y(x)$, it takes time $T[y] = \displaystyle\int_0^a \mathrm{d}x\, \sqrt{\frac{1+y'(x)^2}{2gy(x)}}$ to cover the path-length $S[y] = \displaystyle\int_0^a \mathrm{d}x\, \sqrt{1+y'(x)^2}$, and the average speed is $S[y]/T[y]$.

(a) For the straight-line path, we have $y(x) = bx/a$, which gives $S = \sqrt{a^2+b^2}$ and $T = \sqrt{a^2+b^2}\sqrt{2/(gb)}$; the average speed is $\sqrt{\frac{1}{2}gb} = \sqrt{gR}\sin\dfrac{\phi_0}{2}$.

For the brachistochrone, we have $(\mathrm{d}x)^2 + (\mathrm{d}y)^2 = 2Ry(\mathrm{d}\phi)^2 = \left(2R\sin\frac{\phi}{2}\right)^2(\mathrm{d}\phi)^2$, so that

$$S = \int_0^{\phi_0} \mathrm{d}\phi\, 2R\sin\frac{\phi}{2} = 4R\Bigl(1-\cos\frac{\phi_0}{2}\Bigr) = 8R\Bigl(\sin\frac{\phi_0}{4}\Bigr)^2$$

is the path-length and

$$T = \int_0^{\phi_0} \mathrm{d}\phi\, \sqrt{\frac{2Ry}{2gy}} = \sqrt{\frac{R}{g}}\,\phi_0$$

is the travel time; the average speed is $\sqrt{gR}\,\dfrac{8}{\phi_0}\left(\sin\dfrac{\phi_0}{4}\right)^2$.

The ratio of the two average speeds is

$$\frac{\text{brachistochrone}}{\text{straight line}} = \frac{\dfrac{8}{\phi_0}\Bigl(\sin\dfrac{\phi_0}{4}\Bigr)^2}{2\sin\dfrac{\phi_0}{4}\cos\dfrac{\phi_0}{4}} = \frac{\tan(\phi_0/4)}{\phi_0/4} > 1\,,$$

since $0 < \dfrac{1}{4}\phi_0 < \dfrac{1}{2}\pi$.

(b) As observed in (a), the average speed along a straight-line path with height difference B is $\sqrt{\frac{1}{2}gB}$ if the speed is zero at the upper end, and it will be larger than that if the speed at the upper end is nonzero. We can choose a path that goes on a straight line from $(0,0)$ to an intermediate point (a', B) with $B > b$ and then on another straight line to (a,b), and so get an average speed of $\sqrt{\frac{1}{2}gB}$ or more. Since B can be as large as we like, the average speed can exceed any bound. Conclusion: There is no path for which the average speed is largest.