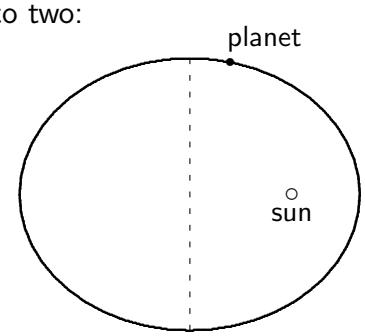
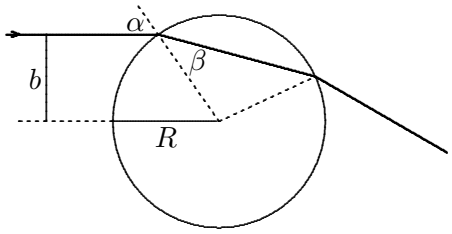
Problem 1 (15 marks)
A planet moves on a Kepler ellipse with numerical eccentricity ϵ. The minor axis of the ellipse cuts the orbit into two:



What fraction of the round-trip time does the planet spend in each half?

Problem 2 (35=10+15+10 marks)
In ray optics the path of a light ray through a ball of water with radius R, without any internal reflection, is as depicted here:



where Snell's law applies in the form $3\sin\alpha = 4\sin\beta$. The ray has impact parameter b and is deflected by angle θ.

(a) Express $y = \cos\left(\frac{1}{2}\theta\right)$ as a function of $x = (b/R)^2$ and sketch the graph of $y(x)$.

(b) Show that the differential cross section is $\dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = -\dfrac{R^2}{8y}\dfrac{\mathrm{d}x}{\mathrm{d}y}$ and find $\dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.

(c) Calculate the total cross section $\sigma = \displaystyle\int \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$. [This is simple!] Do you get the expected result?

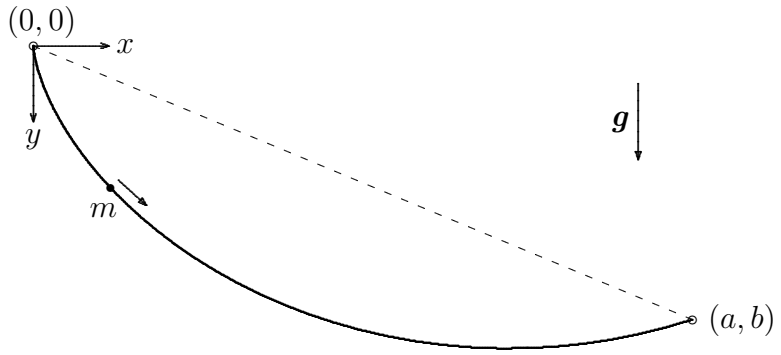
Problem 3 (25=10+15 marks)
Two homogeneous balls, each of mass $\frac{1}{2}M$ and radius R, are centered at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{a}$ with $|\boldsymbol{a}| > 2R$.

(a) What is the quadrupole moment dyadic **Q** of this mass distribution?

(b) The balls are released from rest and then accelerated toward each other by their gravitational attraction. How much time elapses before the balls touch if $|\boldsymbol{a}| \gg R$?

Hint: The integral $\displaystyle\int_0^1 \mathrm{d}x \sqrt{\frac{x}{1-x}} = \frac{1}{2}\pi$ could be useful.

Problem 4 (25=18+7 marks)



A point mass gets from $(x, y) = (0, 0)$ to $(a, b) = R(\phi_0 - \sin\phi_0, 1 - \cos\phi_0)$ along the brachistochrone. It reaches (a, b) in shorter time than it would along the straight line although the path is longer for the brachistochrone. It follows that the average speed for the brachistochrone is larger than that for the straight line.

(a) To confirm this, find the average speed for each path and compare the two speeds.

(b) Is there a path for which the average speed is largest?
Hint: Consider a path composed of straight-line segments.