1 The force

$$F(x) = -V'(x) = E_0 \frac{2a^2 x\,(x^2 - a^2)}{(x^2 + a^2)^3}$$

vanishes at $x = 0$ and $x = \pm a$. At these positions, the potential energy equals $V(x = 0) = 0$ and $V(x = \pm a) = \frac{1}{4}E_0$, and the second derivative of $V(x)$ is

$$V''(x = 0) = -F'(x = 0) = \frac{2E_0}{a^2} \quad \text{and} \quad V''(x = \pm a) = -F'(x = \pm a) = -\frac{E_0}{2a^2}\,.$$

Therefore, we have

$$V(x) \cong \begin{cases} \dfrac{E_0 x^2}{a^2} & \text{for } |x| \ll a\,, \\ \dfrac{E_0\big(a^2 - (x \mp a)^2\big)}{4a^2} & \text{for } |x \mp a| \ll a\,. \end{cases}$$

(a) We have the minimum of the potential energy at $x = 0$ and maxima at $x = \pm a$. It follows (i) that the energy cannot be negative; (ii) that there is motion with no turning points for $E > \frac{1}{4}E_0$; (iii) that, for $0 < E < \frac{1}{4}E_0$, there is motion with two turning points if the initial position is between $x = -a$ and $x = a$, and motion with one turning point otherwise.

(b) We have small-amplitude oscillations near $x = 0$, where $V(x) \cong \frac{1}{2}m\omega_0^2 x^2$ with $\omega_0 = \sqrt{2E_0/(ma^2)}$, so that

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi a}{\sqrt{2E_0/m}}$$

is their period.

(c) When $E_0 = -|E_0| < 0$, the potential energy has its maximum at $x = 0$ and two symmetric minima at $x = \pm a$. Then there is (i) motion with no turning point for $E > 0$, and (ii) motion with two turning points for $\frac{1}{4}E_0 < E < 0$. We have small-amplitude oscillations near $x = a$ and $x = -a$ with the same period $T_1 = 2\pi/\omega_1$ with ω_1 given by $V(x) \cong \frac{E_0}{4a^2} + \frac{1}{2}m\omega_1^2(x \mp a)^2$, so that $\omega_1^2 = -E_0/(2ma^2)$ and

$$T_1 = \frac{4\pi a}{\sqrt{2|E_0|/m}}$$

is the corresponding period.

2

(a) At the perihelion we have $s = (1-\epsilon)a$, $\dot{s} = 0$, and $\dot{\varphi} = \kappa/s^2$ as always, so that

$$E = \frac{m}{2}\big(\dot{s}^2 + (s\dot{\varphi})^2\big) - \frac{Gm_\odot m}{s} = \frac{m\kappa^2}{2(1-\epsilon)^2a^2} - \frac{Gm_\odot m}{(1-\epsilon)a}\,.$$

(b) The virial theorem of Exercises 31–33 applies here for $n = -1$, so that

$$\overline{E_{\rm kin}} = -E \quad \text{and} \quad \overline{E_{\rm pot}} = 2E\,.$$

(c) With $\mathrm{d}t = \mathrm{d}\varphi\, s^2/\kappa$, we have

$$\begin{aligned}\overline{E_{\rm pot}} &= \frac{1}{T}\int_0^T \mathrm{d}t\,\frac{-Gm_\odot m}{s} = -\frac{Gm_\odot m}{\kappa T}\int_0^{2\pi}\mathrm{d}\varphi\, s(\varphi)\\ &= -\frac{Gm_\odot m}{\kappa T}\int_0^{2\pi}\mathrm{d}\varphi\,\frac{(1-\epsilon^2)a}{1+\epsilon\cos\varphi} = -\frac{Gm_\odot m}{\kappa T}2\pi a\sqrt{1-\epsilon^2}\\ &= -\frac{Gm_\odot m}{a}\,.\end{aligned}$$

(d) We equate $2E$ from (a) with $\overline{E_{\rm pot}}$ from (c), and solve for $Gm_\odot$. This gives

$$Gm_\odot = \frac{\kappa^2}{(1-\epsilon^2)a} = \left(\frac{2\pi}{T}\right)^2 a^3\left(\frac{\kappa T}{2\pi\sqrt{1-\epsilon^2}a^2}\right)^2 = \left(\frac{2\pi}{T}\right)^2 a^3\,,$$

which is Kepler's Third Law.

3

(a) When $|\boldsymbol{v}| \ll c$, we have $\sqrt{c^2 - \boldsymbol{v}^2} = c - \frac{1}{2c}\boldsymbol{v}^2$, so that

$$L = mc^2 - mc\left(c - \frac{1}{2c}\boldsymbol{v}^2\right) - V(\boldsymbol{r}) = \frac{1}{2}m\boldsymbol{v}^2 - V(\boldsymbol{r})\,,$$

as it should be.

(b) The momentum is related to the velocity by

$$\boldsymbol{p} = \boldsymbol{\nabla}_v L = \frac{mc\boldsymbol{v}}{\sqrt{c^2 - \boldsymbol{v}^2}}\,.$$

We square this to establish first

$$c^2 - \boldsymbol{v}^2 = \frac{(mc^2)^2}{(mc)^2 + \boldsymbol{p}^2}$$

and then

$$\boldsymbol{v} = \frac{c\boldsymbol{p}}{\sqrt{(mc)^2 + \boldsymbol{p}^2}}\,.$$

It follows that the Hamilton function is

$$H = \Bigl(\boldsymbol{p}\cdot\boldsymbol{v} - L\Bigr)\Bigg|_{\boldsymbol{v}\,=\,\text{(as above)}} = c\sqrt{(mc)^2 + \boldsymbol{p}^2} - mc^2 + V(\boldsymbol{r})\,.$$

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4 The top mass has these coordinates and velocity components:

$$\begin{aligned}(x_1, y_1) &= R(\phi + \sin\phi, 1 - \cos\phi)\\ (\dot{x}_1, \dot{y}_1) &= R\dot{\phi}(1 + \cos\phi, \sin\phi)\,;\end{aligned}$$

and for the bottom mass we have

$$\begin{aligned}(x_2, y_2) &= (x_1, y_1) + 3R(\sin\theta, -\cos\theta)\\ (\dot{x}_2, \dot{y}_2) &= (\dot{x}_1, \dot{y}_1) + 3R\dot{\theta}(\cos\theta, \sin\theta)\,.\end{aligned}$$

(a) In the Lagrange function $L = E_{\text{kin}} - E_{\text{pot}}$, we have the kinetic energy

$$\begin{aligned}E_{\text{kin}} &= \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2)\\ &= 2mR^2\dot{\phi}^2(1 + \cos\phi) + 3mR^2\dot{\phi}\dot{\theta}\bigl(\cos\theta + \cos(\phi - \theta)\bigr) + \frac{9}{2}mR^2\dot{\theta}^2\end{aligned}$$

and the potential energy

$$E_{\text{pot}} = mgy_1 + mg(y_2 + 3R) = mgR(5 - 2\cos\phi - 3\cos\theta)\,,$$

where we recognize that $y_1 = 0$, $y_2 = -3R$ at equilibrium and choose to set $E_{\text{pot}} = 0$ there.

(b) Near this equilibrium we have

$$E_{\text{kin}} = 4mR^2\dot{\phi}^2 + 6mR^2\dot{\phi}\dot{\theta} + \frac{9}{2}mR^2\dot{\theta}^2\,,$$
$$E_{\text{pot}} = mgR\Big(\phi^2 + \frac{3}{2}\theta^2\Big)\,,$$

which give

$$L = \frac{1}{2}\,(\,\dot{\phi}\;\dot{\theta}\,)\,M\begin{pmatrix}\dot{\phi}\\ \dot{\theta}\end{pmatrix} - \frac{1}{2}\,(\,\phi\;\theta\,)\,K\begin{pmatrix}\phi\\ \theta\end{pmatrix}$$

with

$$M = mR^2\begin{pmatrix}8 & 6\\ 6 & 9\end{pmatrix} \quad\text{and}\quad K = mgR\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\,.$$

(c) After putting the common factor mR aside, the frequencies of the normal modes are determined by

$$\det\left\{\begin{pmatrix}8R\omega^2 - 2g & 6R\omega^2\\ 6R\omega^2 & 9R\omega^2 - 3g\end{pmatrix}\right\} = 36(R\omega^2)^2 - 42gR\omega^2 + 6g^2 = 0$$

or

$$(R\omega^2 - g)(6R\omega^2 - g) = 0\,,$$

so that the normal frequencies are $\omega_1 = \sqrt{g/R}$ and $\omega_2 = \sqrt{g/(6R)}$. The corresponding normal coordinates follow from

$$\begin{pmatrix}8R\omega^2 - 2g & 6R\omega^2\\ 6R\omega^2 & 9R\omega^2 - 3g\end{pmatrix}_{R\omega^2 = g}\begin{pmatrix}\phi_1\\ \theta_1\end{pmatrix} = 0 \quad\text{or}\quad \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \theta_1\end{pmatrix} = 0$$

and

$$\begin{pmatrix}8R\omega^2 - 2g & 6R\omega^2\\ 6R\omega^2 & 9R\omega^2 - 3g\end{pmatrix}_{R\omega^2 = g/6}\begin{pmatrix}\phi_2\\ \theta_2\end{pmatrix} = 0 \quad\text{or}\quad \begin{pmatrix}-4 & 6\\ 6 & -9\end{pmatrix}\begin{pmatrix}\phi_2\\ \theta_2\end{pmatrix} = 0\,.$$

Accordingly, we can choose $\begin{pmatrix}\phi_1\\ \theta_1\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}$ and $\begin{pmatrix}\phi_2\\ \theta_2\end{pmatrix} = \begin{pmatrix}3\\ 2\end{pmatrix}$.

A small-amplitude oscillation in the fast normal mode ($\omega = \omega_1$) is of the form

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}2R\phi\\ 2R\phi + 3R\theta\end{pmatrix}\Bigg|_{\phi = -\theta = \epsilon_1} = \begin{pmatrix}2R\epsilon_1\\ -R\epsilon_1\end{pmatrix}$$

and $(y_1, y_2) = (0, -3R)$, where $\epsilon_1(t) = a_1\cos(\omega_1 t - \varphi_1)$ with some small amplitude a_1 and some phase φ_1. The two masses are displaced to opposite sides, whereby the top mass is oscillating with twice the amplitude of the bottom mass.

Likewise, a small-amplitude oscillation in the slow normal mode ($\omega = \omega_2$) is of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2R\phi \\ 2R\phi + 3R\theta \end{pmatrix}\Bigg|_{2\phi = 3\theta = \epsilon_2} = \begin{pmatrix} R\epsilon_2 \\ 2R\epsilon_2 \end{pmatrix}$$

and $(y_1, y_2) = (0, -3R)$, where with $\epsilon_2(t) = a_2 \cos(\omega_2 t - \varphi_2)$ with some small amplitude a_2 and some phase φ_2. The two masses are displaced to the same side, whereby the top mass is oscillating with half the amplitude of the bottom mass.
