1. One-dimensional motion (25=9+6+10 marks)

A point mass m is moving along the x axis under the influence of the force associated with the potential energy

$$V(x) = E_0 \frac{a^2 x^2}{(x^2 + a^2)^2} \quad \text{with} \quad E_0 > 0 \quad \text{and} \quad a > 0.$$

(a) For which energy ranges do you have motion with one, two, or no turning points?

(b) For the oscillatory motion between two turning points, what is the period of small-amplitude oscillations?

(c) Answer the questions of (a) and (b) for $E_0 < 0$.

2. Sun and planet (30=8+4+10+8 marks)

A planet of mass m moves on a Kepler ellipse with major half-axis a, numerical eccentricity ϵ, angular momentum $|\boldsymbol{l}| = m\kappa > 0$, and period $T = \dfrac{2\pi}{\kappa} a^2 \sqrt{1 - \epsilon^2}$.

(a) Express the energy as the sum of the kinetic energy and the potential energy when the planet is in its perihelion.

(b) What does the virial theorem say about the time averages of the kinetic and the potential energy, averaged over one period?

(c) Find the time average of the potential energy by an explicit integration.

(d) Derive Kepler's Third Law by combining (a), (b), and (c).

Hint: The integral $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\alpha}{A + B\cos\alpha} = \frac{2\pi}{\sqrt{A^2 - B^2}}$ for $A > B > 0$ could be useful.

3. Relativistic motion (15=5+10 marks)

The Lagrange function for a particle of mass m in relativistic motion is

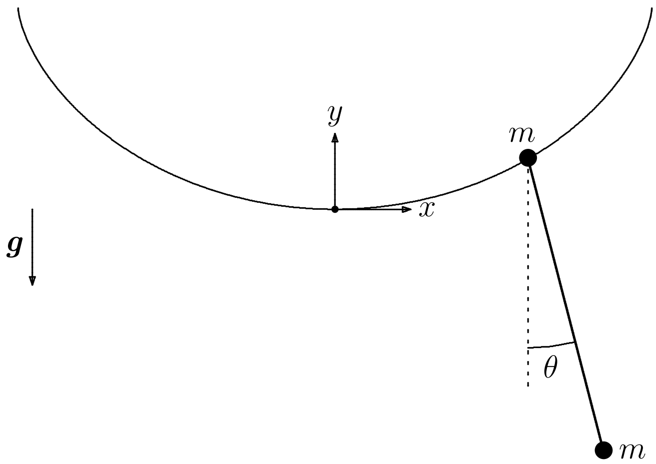
$$L = mc^2 - mc\sqrt{c^2 - \boldsymbol{v}^2} - V(\boldsymbol{r}),$$

where c is the speed of light and $V(\boldsymbol{r})$ is the potential energy.

(a) Show that this is the familiar nonrelativistic expression when $|\boldsymbol{v}| \ll c$.

(b) What is the corresponding Hamilton function?

4. Normal modes (30=10+8+12 marks)



Two equal point masses m are moving without friction in the vertical xy plane, with the gravitational acceleration $\boldsymbol{g} = -g\boldsymbol{e}_y$. The top mass moves along the cycloid parameterized by $(x, y) = R(\phi + \sin\phi, 1 - \cos\phi)$ with $-\pi < \phi < \pi$ and $R > 0$. The bottom mass is connected to the top mass by a massless string of length $3R$; the string is always fully stretched and has angle θ with the vertical direction.

(a) State the Lagrange function $L(\phi, \dot{\phi}, \theta, \dot{\theta})$.

(b) What is the approximate Lagrange function for small-amplitude oscillations around the equilibrium configuration?

(c) Find the normal modes and describe them.