1

(a) On the way up, the weight and the frictional force are both downward; on the way down, the weight is downward but the frictional force is upward. Therefore, there is a larger net force between $t = 0$ and $t = t_1$ than between $t = t_1$ and $t = t_2$, with a correspondingly larger acceleration. To cover the same distance, then, it takes a shorter time upwards than downwards.

(b) The differential equation

$$\ddot{z}(t) = -g - \gamma \dot{z}(t)$$

with $\dot{z}(t=0) = v_0$ and $z(t=0) = 0$ is solved by

$$\begin{aligned} \dot{z}(t) &= -\frac{g}{\gamma} + \Bigl(v_0 + \frac{g}{\gamma}\Bigr)\,\mathrm{e}^{-\gamma t}\,, \\ z(t) &= -\frac{gt}{\gamma} + \Bigl(v_0 + \frac{g}{\gamma}\Bigr)\frac{1 - \mathrm{e}^{-\gamma t}}{\gamma}\,. \end{aligned}$$

From $\dot{z}(t_1) = 0$ and $z(t_2) = 0$, we get

$$1 + \frac{\gamma v_0}{g} = \mathrm{e}^{\gamma t_1} = \frac{\gamma t_2}{1 - \mathrm{e}^{-\gamma t_2}}\,,$$

so that

$$\mathrm{e}^{\gamma t_1} = \mathrm{e}^{\frac{1}{2}\gamma t_2}\frac{\frac{1}{2}\gamma t_2}{\sinh\bigl(\frac{1}{2}\gamma t_2\bigr)} < \mathrm{e}^{\frac{1}{2}\gamma t_2}\,,$$

and $t_2 > 2t_1$ follows.

2

(a) a is a length, and F is a force.

(b) Fa is an energy, and $\sqrt{ma/F}$ is a time.

(c) We have

$$\sqrt{|x| + a} - \sqrt{a} \cong \begin{cases} \frac{1}{2}\sqrt{x^2/a} & \text{for} \quad |x| \ll a\,, \\ \sqrt{|x|} & \text{for} \quad |x| \gg a\,, \end{cases}$$

so that

$$V(x) \cong \begin{cases} \dfrac{1}{4a}Fx^2 & \text{for} \quad |x| \ll a\,, \\ F|x| & \text{for} \quad |x| \gg a\,. \end{cases}$$

(d) For $\frac{1}{2}m\omega^2 x^2 = \frac{1}{4a}Fx^2$ and $T = \frac{2\pi}{\omega}$, we get

$$T = 2\pi\sqrt{\frac{2ma}{F}}\,.$$

(e) The turning points are at $x = \pm x_1$ with $E = V(x_1)$. Then

$$T(E) = 2\int\limits_{-x_1}^{x_1} \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}\big[V(x_1) - V(x)\big]}} = 4\int\limits_{0}^{x_1} \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}\big[V(x_1) - V(x)\big]}}\,.$$

We substitute $x = (y^2 + 2y)a$, $\mathrm{d}x = \mathrm{d}y\, 2(y+1)a$, $V(x) = Fay^2$, $E = V(x_1) = Fay_1^2$ and get

$$T(E) = 4\int\limits_{0}^{y_1} \frac{\mathrm{d}y\, 2(y+1)a}{\sqrt{\frac{2}{m}Fa(y_1^2 - y^2)}} = 4\sqrt{\frac{2ma}{F}}\int\limits_{0}^{y_1} \frac{\mathrm{d}y\,(y+1)}{\sqrt{y_1^2 - y^2}}\,.$$

With

$$\int\limits_{0}^{y_1} \frac{\mathrm{d}y\, y}{\sqrt{y_1^2 - y^2}} = -\sqrt{y_1^2 - y^2}\,\Bigg|_{y=0}^{y_1} = y_1$$

and

$$\int\limits_{0}^{y_1} \frac{\mathrm{d}y}{\sqrt{y_1^2 - y^2}} = \frac{1}{2}\pi\,,$$

this gives

$$T(E) = \sqrt{\frac{2ma}{F}}\left(4\sqrt{\frac{E}{Fa}} + 2\pi\right).$$

3

(a) Since

$$\boldsymbol{\nabla} \times \boldsymbol{F} \,\widehat{=}\, \begin{pmatrix} 4\lambda z \\ 0 \\ 0 \end{pmatrix} \neq 0\,,$$

this force is not conservative.

(b) Since

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{\nabla} \times \left(a^2 \boldsymbol{r} - \boldsymbol{a}\, \boldsymbol{a} \cdot \boldsymbol{r}\right) = a^2 \underbrace{\boldsymbol{\nabla} \times \boldsymbol{r}}_{=0} + \boldsymbol{a} \times \underbrace{\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{r})}_{=\boldsymbol{a}} = 0\,,$$

this force is conservative, and the potential energy is

$$V(\boldsymbol{r}) = -\frac{1}{2}a^2 r^2 + \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{r})^2 = -\frac{1}{2}(\boldsymbol{a} \times \boldsymbol{r})^2\,.$$
