1

(a) We have

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 \cos(\omega_0 t) + \frac{\boldsymbol{v}_0}{\omega_0} \sin(\omega_0 t)$$

as well as

$$E = \frac{1}{2} m \boldsymbol{v}_0^2 + \frac{1}{2} m \omega_0^2 \boldsymbol{r}_0^2 \quad \text{and} \quad \boldsymbol{l} = m \boldsymbol{r}_0 \times \boldsymbol{v}_0 \,.$$

(b) With $\dot{\boldsymbol{v}} = -\omega_0^2 \boldsymbol{r}$ and $\dot{\boldsymbol{r}} = \boldsymbol{v}$, the time derivative of D is

$$\begin{aligned} \dot{\mathsf{D}} &= (\dot{\boldsymbol{v}}\, \boldsymbol{v} + \boldsymbol{v}\, \dot{\boldsymbol{v}}) + \omega_0^2 (\dot{\boldsymbol{r}}\, \boldsymbol{r} + \boldsymbol{r}\, \dot{\boldsymbol{r}}) \\ &= -\omega_0^2 (\boldsymbol{r}\, \boldsymbol{v} + \boldsymbol{v}\, \boldsymbol{r}) + \omega_0^2 (\boldsymbol{v}\, \boldsymbol{r} + \boldsymbol{r}\, \boldsymbol{v}) = 0 \,. \end{aligned}$$

(c) Since

$$(\boldsymbol{l} \times \boldsymbol{r}_0) \cdot \boldsymbol{r}(t) = (\boldsymbol{l} \times \boldsymbol{r}_0) \cdot \frac{\boldsymbol{v}_0}{\omega_0} \sin(\omega_0 t) = \frac{1}{m \omega_0} \boldsymbol{l}^2 \sin(\omega_0 t)$$

and

$$(\boldsymbol{v}_0 \times \boldsymbol{l}) \cdot \boldsymbol{r}(t) = (\boldsymbol{v}_0 \times \boldsymbol{l}) \cdot \boldsymbol{r}_0 \cos(\omega_0 t) = \frac{1}{m} \boldsymbol{l}^2 \cos(\omega_0 t) \,,$$

we have

$$\boldsymbol{r}(t) \cdot [\boldsymbol{a}_1\, \boldsymbol{a}_1 + \boldsymbol{a}_2\, \boldsymbol{a}_2] \cdot \boldsymbol{r}(t) = 1 \,,$$

where

$$\boldsymbol{a}_1 = \frac{m \omega_0 \boldsymbol{l} \times \boldsymbol{r}_0}{\boldsymbol{l}^2} \quad \text{and} \quad \boldsymbol{a}_2 = \frac{m \boldsymbol{v}_0 \times \boldsymbol{l}}{\boldsymbol{l}^2}$$

are non-parallel vectors in the plane of motion. If we choose that to be the xy plane, then $\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} = 1$ with a positive, symmetric 2×2 matrix A.

2

(a) We have a circular orbit if E equals the minimum value of the effective potential energy $V_{\text{eff}}(s) = V(s) + \dfrac{m\kappa^2}{2s^2}$, so this E is the smallest possible energy for the given κ. In return, larger κ values are not possible for this E, because then the minimum value of $V_{\text{eff}}(s)$ would be larger, too.

(b) Here, we have

$$V_{\text{eff}}(s) = -\frac{A}{s(a+s)^2} + \frac{m\kappa^2}{2s^2} = \frac{1}{2s^2(a+s)^2} \left[m\kappa^2 (a+s)^2 - 2As \right] ,$$

which is positive for sufficiently small distances s and for sufficiently large distances s. Therefore, orbits with $E = 0$ are possible only if the minimum value of the factor $[\cdots]$ is negative. This is the case if

$$[\cdots] \Big|_{m\kappa^2(s+a) = A} = 2Aa - \frac{A^2}{m\kappa^2} < 0 \quad \text{or} \quad \kappa^2 < \frac{A}{2ma} \,.$$

(c) For

$$V_{\text{eff}}(s) = -\frac{m\kappa^2}{2s^2(a+s)^2}(s_2 - s)(s - s_1)$$

with

$$s_s s_2 = a^2 \quad \text{and} \quad s_1 + s_2 = \frac{2A}{m\kappa^2} - 2a > 2a\,,$$

we get

$$\begin{aligned}\Phi &= 2\int_{s_1}^{s_2} \frac{\mathrm{d}s}{s} \frac{\kappa}{\sqrt{-\frac{2}{m}s^2 V_{\text{eff}}(s)}} = 2\int_{s_1}^{s_2} \frac{\mathrm{d}s}{s} \frac{a+s}{\sqrt{(s_2 - s)(s - s_1)}} \\ &= \frac{2\pi a}{\sqrt{s_1 s_2}} + 2\pi = 4\pi\end{aligned}$$

after using (5.3.21), (5.3.27), and (3.1.40).

3

(a) From Exercise 37, we know that

$$\tan\Theta = \frac{\sin\theta}{m_1/m_2 + \cos\theta}\,.$$

(i) When $m_1 < m_2$, the denominator vanishes when $\cos\theta = -m_1/m_2 > -1$ and is negative for larger θ values. It follows that all values in the range $0 \le \Theta \le \pi$ are possible.
(ii) When $m_1 = m_2$, we have $\tan\Theta = \tan(\frac{1}{2}\theta)$ or $\Theta = \frac{1}{2}\theta$, and all values in the range $0 \le \Theta \le \frac{1}{2}\pi$ are possible.
(iii) When $m_1 > m_2$, the denominator is always positive, and $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\tan\Theta = 0$ when $m_1\cos\theta = -m_2$, so that the largest possible value for $\tan\Theta$ is $m_2/\sqrt{m_1^2 - m_2^2}$, and the possible Θ values are from the range $0 \le \Theta \le \sin^{-1}(m_2/m_1) < \frac{1}{2}\pi$.

(b) Corresponding cross sections are equal,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} 2\pi \mathrm{d}\theta\,\sin\theta = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{lab}} 2\pi\mathrm{d}\Theta\,\sin\Theta\,,$$

so that

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{lab}} = f(\theta)\frac{\sin\theta}{\sin\Theta}\frac{\mathrm{d}\theta}{\mathrm{d}\Theta}\bigg|_{\theta\,=\,2\Theta} = 4f(2\Theta)\cos\Theta\,.$$
