Problem 1 (35=10+10+15 marks)
Isotropic harmonic oscillator: Point mass m moves in the force field $\boldsymbol{F} = -m\omega_0^2\boldsymbol{r} = -\boldsymbol{\nabla}\frac{1}{2}m\omega_0^2r^2$ with energy E and angular momentum $\boldsymbol{l} \neq 0$.

(a) By generalizing the familiar $x(t)$ of the one-dimensional harmonic oscillator, state $\boldsymbol{r}(t)$ for initial position $\boldsymbol{r}(t=0) = \boldsymbol{r}_0$ and initial velocity $\boldsymbol{v}(t=0) = \boldsymbol{v}_0$. Express E and $\boldsymbol{l}$ in terms of $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$.

(b) Show that the dyadic $\mathbf{D} = \boldsymbol{v}\,\boldsymbol{v} + \omega_0^2\boldsymbol{r}\,\boldsymbol{r}$ does not depend on time.

(c) Consider $(\boldsymbol{l} \times \boldsymbol{r}_0) \cdot \boldsymbol{r}(t)$ and $(\boldsymbol{v}_0 \times \boldsymbol{l}) \cdot \boldsymbol{r}(t)$ and use them to show that the plane orbit is an ellipse centered at $\boldsymbol{r} = 0$. — Hint: $x(t), y(t)$ trace out a centered ellipse if $(x\ y)A\begin{pmatrix} x \\ y \end{pmatrix} = 1$ with a positive, symmetric 2×2 matrix A.

Problem 2 (40=10+15+15 marks)
Point mass m moves in the central-force field associated with the potential energy $V(r)$. The force is attractive, $V'(r) > 0$, and the motion is confined to the radial range $s_1 \leq r \leq s_2$. As usual the bounds are determined by the energy E and the angular momentum $\boldsymbol{l}$ of the orbit.

(a) Circular orbits ($s_1 = s_2$) have (i) smallest energy for given angular momentum and (ii) largest angular momentum for given energy. Explain why this is so.

(b) For the potential energy $V(r) = -\dfrac{A}{r(a+r)^2}$ with constants $A > 0$ and $a > 0$, one can have bound orbits for $E = 0$. For which values of $\kappa = |\boldsymbol{l}|/m$ is this possible?

(c) Find the angular period of such an orbit with $E = 0$.

Problem 3 (25=15+10 marks)
A projectile with mass m_1 is scattered by a target of mass m_2, whereby a conservative line-of-sight force is acting. The target is at rest before the scattering. The scattering angle in the center-of-mass frame is denoted by θ, that in the laboratory frame by Θ.

(a) What is the range of possible Θ values when (i) $m_1 < m_2$, (ii) $m_1 = m_2$, (iii) $m_1 > m_2$?

(b) If the differential cross section in the center-of-mass frame is $\dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = f(\theta)$, what is the differential cross section observed in the laboratory frame when $m_1 = m_2$?