1. One-dimensional motion with friction (20=10+10 marks)

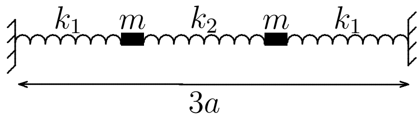
A stone ($\equiv$ point mass m) is thrown upward from ground level with initial velocity $\boldsymbol{v}(t=0) = v_0\boldsymbol{e}_z$, $v_0 > 0$, and then moves under the influence of its weight $m\boldsymbol{g} = -mg\boldsymbol{e}_z$ and the air-drag frictional force $-m\kappa v\boldsymbol{v} = -mgv\boldsymbol{v}/v_\infty^2$ with $v_\infty = \sqrt{g/\kappa} > 0$.

(a) What height h above ground does the stone reach?

(b) When the stone is back at ground level, what is its speed v_1? Express v_1 in terms of v_0 and v_∞.

Hint: Recall that $\tan\alpha = -\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}\alpha}\log\Big((\cos\alpha)^2\Big)$, $\tanh\alpha = \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}\alpha}\log\Big((\cosh\alpha)^2\Big)$.

2. Normal modes (30=10+10+5+5 marks)



Two point masses m are coupled to three springs as shown in the figure, where k_1 and k_2 are the spring constants for the outer springs and the inner spring, respectively. Each spring has length a when it is relaxed. The masses of the springs are negligibly small. The two point masses can only move along the horizontal line specified by the figure, and no forces other than those of the springs are acting.

(a) Choose suitable coordinates and state the Lagrange function for this system.

(b) Find the normal modes and their characteristic frequencies.

(c) Use words and suitable sketches to describe the normal modes.

(d) What is the Hamilton function associated with your Lagrange function?

3. Moments of inertia (20=8+7+5 marks)

(a) Body 1 has mass M_1 and inertia dyadic $\mathbf{I}_1$; likewise, there are M_2 and $\mathbf{I}_2$ for body 2. If the center-of-mass of body 1 is at position $\boldsymbol{R}_1$ and that of body 2 at $\boldsymbol{R}_2$, what is the inertia dyadic $\mathbf{I}$ of this two-body system?

(b) Four equal point masses m are placed at the non-adjacent corners of a cube with volume a^3. What is the inertia dyadic of this four-mass system?

(c) One of the four masses is removed. What is the inertia dyadic of the remaining three-mass system? State your answer in terms of $\boldsymbol{r}_4$, the vector from the center of the cube to the corner of the removed fourth mass.

4. Stabilization by a periodic force (30=8+12+10 marks)

A point mass m is near $\boldsymbol{r} = 0$, where we have the center of a ring with a large diameter; the ring and the point mass carry electric charge of the same sign, so that the force on the point mass derives from the potential energy

$$V(\boldsymbol{r}) = \frac{1}{2} m\omega_0^2 \left[r^2 - 3(\boldsymbol{n} \cdot \boldsymbol{r})^2 \right] ,$$

where $\boldsymbol{n}$ is the unit vector normal to the plane of the ring. The ring rotates about a diameter with angular velocity Ω, so that

$$\boldsymbol{n} = \boldsymbol{e}_x \cos(\Omega t) + \boldsymbol{e}_y \sin(\Omega t) ,$$

if we choose the z axis as the axis of rotation and have the ring in the yz plane at $t = 0$.

(a) What is the equation of motion for the point mass in the laboratory frame?

(b) What is the equation of motion for the point mass in the rotating frame in which the ring does not move?

(c) How do we need to choose Ω to ensure that the point mass remains near $\boldsymbol{r} = 0$?

Hint: Try an exponential ansatz.