$\boxed{1}$ The number of bosons that can be accommodated in excited states is

$$N - N_0 = \frac{L}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}p\,\frac{z}{\mathrm{e}^{\beta p^2/(2m)} - z} = \frac{L}{2\pi\hbar}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}p\left(z\mathrm{e}^{-\beta p^2/(2m)}\right)^k$$
$$= \frac{L}{\lambda_\beta}\sum_{k=1}^{\infty}\frac{z^k}{k^{1/2}} = \frac{L}{\lambda_\beta}g_{1/2}(z)\,,$$

where $\lambda_\beta = \hbar\sqrt{2\pi\beta/m}$ is the thermal wavelength and the fugacity z is from the range $0 < z < 1$. Since $g_{1/2}(z) \to \infty$ as $z \to 1$, we can have any number of bosons in the excited states and, therefore, there will not be an excess of bosons in the ground state. It follows that there is no Bose-Einstein condensation here.

$\boxed{2}$

(a) $E_{\mathrm{kin}}[\rho]$ is the usual TF approximation for the kinetic energy; $E_{\mathrm{trap}}[\rho]$ is simply obtained from the general form for the energy in an external potential, $\int(\mathrm{d}\vec{r})\,V_{\mathrm{ext}}(\vec{r})\rho(\vec{r})$, with the potential energy of the harmonic force for the external potential energy $V_{\mathrm{ext}}(\vec{r})$; and $E_{\mathrm{int}}[\rho]$ is likewise obtained from the general expression for the pair-interaction energy, $E_{\mathrm{int}}[\rho] = \frac{1}{2}\int(\mathrm{d}\vec{r}_1)\int(\mathrm{d}\vec{r}_2)\,\rho(\vec{r}_1)V_{\mathrm{int}}(\vec{r}_1 - \vec{r}_2)\rho(\vec{r}_2)$, with the interaction potential $V_{\mathrm{int}}(\vec{r}) = W\delta(\vec{r})$

(b) There is first the constraint $\int(\mathrm{d}\vec{r})\rho(\vec{r}) = N$, and then we get from $\delta E[\rho] = 0$ that

$$\frac{\hbar^2}{2m}[3\pi^2\rho(\vec{r})]^{2/3} + \frac{1}{2}m\omega^2\vec{r}^{\,2} + W\rho(\vec{r}) = \mu$$

must hold where $\rho \neq 0$, with the chemical potential μ. Its value is determined by the constraint.

(c) The trial densities $\rho_\lambda(\vec{r}) = \lambda^3\rho(\lambda\vec{r})$ obey the constraint and give

$$E[\rho_\lambda] = \lambda^2 E_{\mathrm{kin}}[\rho_{\mathrm{TF}}] + \lambda^{-2}E_{\mathrm{trap}}[\rho_{\mathrm{TF}}] + \lambda^3 E_{\mathrm{int}}[\rho_{\mathrm{TF}}]\,.$$

Then, $\frac{\partial}{\partial\lambda}E[\rho_\lambda]\Big|_{\lambda=1} = 0$ implies $2E_{\mathrm{kin}}[\rho_{\mathrm{TF}}] - 2E_{\mathrm{trap}}[\rho_{\mathrm{TF}}] + 3E_{\mathrm{int}}[\rho_{\mathrm{TF}}] = 0$.

$\boxed{3}$

(a) With $Q(\beta, V, N) = \dfrac{1}{N!}\left(\dfrac{V}{\pi^2(\hbar c\beta)^3}\right)^N$ we get

$$Z(\beta, V, z) = \sum_{N=0}^{\infty} Q(\beta, V, N) z^N = \exp\left(\frac{zV}{\pi^2(\hbar c\beta)^3}\right).$$

(b) We have

$$F(T, V, N) = -k_{\mathrm{B}}T \log Q(\beta, V, N) = -Nk_{\mathrm{B}}T\left[1 + \log\frac{(k_{\mathrm{B}}T)^3 V/N}{\pi^2(\hbar c)^3}\right]$$

after using Stirling's formula for $\log N!$ as usual. It follows that the pressure is $p = -\dfrac{\partial F}{\partial V} = Nk_{\mathrm{B}}T/V$ and, therefore,

$$G(T, p, N) = \Big(F(T, V, N) + pV\Big)_{V = Nk_{\mathrm{B}}T/p} = -Nk_{\mathrm{B}}T \log\frac{(k_{\mathrm{B}}T)^4}{\pi^2(\hbar c)^3 p}.$$

(c) Since $S(T, V, N) = -\dfrac{\partial F(T, V, N)}{\partial T}$ and $S(T, p, N) = -\dfrac{\partial G(T, p, N)}{\partial T}$, the heat capacitances are

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = -T\left(\frac{\partial}{\partial T}\right)^2 F(T, V, N) = 3k_{\mathrm{B}}N$$

and

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_{p,N} = -T\left(\frac{\partial}{\partial T}\right)^2 G(T, p, N) = 4k_{\mathrm{B}}N.$$