Problem 1 (**30** marks)
An ideal gas of bosons is confined to a line of length L. Does Bose-Einstein condensation occur in such a one-dimensional boson gas? If yes, what is the critical temperature for given length L?

Problem 2 (**30**=6+12+12 marks)
An ultracold (temperature $T = 0$) gas of N spin-$\frac{1}{2}$ atoms of mass m is trapped by a harmonic force with potential energy $\frac{1}{2}m\omega^2\vec{r}^{\,2}$. The atoms interact with a repulsive contact force, so that the potential energy is $W\delta(\vec{r}_1 - \vec{r}_2)$ for one atom at $\vec{r}_1$ and another at $\vec{r}_2$, where $W > 0$.

(a) Explain why the energy functional of the atom density $\rho(\vec{r})$ is

$$E[\rho] = \int(\mathrm{d}\vec{r})\,\frac{\hbar^2}{10\pi^2 m}[3\pi^2\rho(\vec{r})]^{5/3} + \int(\mathrm{d}\vec{r})\,\frac{1}{2}m\omega^2\vec{r}^{\,2}\rho(\vec{r}) + \frac{1}{2}\int(\mathrm{d}\vec{r})\,W\rho(\vec{r})^2$$

$$\equiv E_{\mathrm{kin}}[\rho] + E_{\mathrm{trap}}[\rho] + E_{\mathrm{int}}[\rho]$$

in the Thomas-Fermi approximation.

(b) Which equations are obeyed by the density $\rho_{\mathrm{TF}}(\vec{r})$ for which $E[\rho]$ is minimal?

(c) Show that $2E_{\mathrm{kin}}[\rho_{\mathrm{TF}}] - 2E_{\mathrm{trap}}[\rho_{\mathrm{TF}}] + 3E_{\mathrm{int}}[\rho_{\mathrm{TF}}] = 0$.

Problem 3 (**40**=10+15+15 marks)
An ideal gas of luxons — massless, conserved particles with kinetic energy $c|\vec{p}|$, where c is the speed of light — is confined to volume V.

(a) In an Exercise, you found the canonical partition function $Q(\beta, V, N)$. What is the grand-canonical partition function $Z(\beta, V, z)$?

(b) Find the Helmholtz free energy $F(T, V, N)$ and the Gibbs free energy $G(T, p, N)$ as functions of their natural variables.

(c) Determine the heat capacitances C_V and C_p for constant volume and constant pressure, respectively.