1

(a) The equation of state says $S\frac{\partial U}{\partial S} = \alpha L\frac{\partial U}{\partial L}$, which implies that U is a function of $S^\alpha L$, and since S, L, n and U are extensive, we conclude that $U(S, L, n) = nf(S^\alpha L/n^{\alpha+1})$.

(b) The analog is $S\mathrm{d}T + L\mathrm{d}\tau + n\mathrm{d}\mu = 0$ or $\mathrm{d}\mu = -(S/n)\mathrm{d}T - (L/n)\mathrm{d}\tau$.

(c) From $U = \gamma(S/n)^\alpha L$, we find

$$T = \alpha\gamma s^{\alpha-1}\ell\,, \quad \tau = \gamma s^\alpha\,, \quad \mu = -\alpha\gamma s^\alpha \ell\,,$$

where $s = S/n$ and $\ell = L/n$, so that

$$\mu(T, \tau) = -T\left(\frac{\tau}{\gamma}\right)^{1/\alpha} \quad \text{and} \quad \mathrm{d}\mu = \frac{\mu}{T}\mathrm{d}T + \frac{\mu}{\alpha\tau}\mathrm{d}\tau\,.$$

In view of $\mu/T = -s$ and $\mu/(\alpha\tau) = -\ell$, we have $\mathrm{d}\mu = -s\mathrm{d}T - \ell\mathrm{d}\tau$, indeed.

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2 The number of bosons that can be accommodated in excited states is

$$\begin{aligned} N - N_0 &= \frac{A}{(2\pi\hbar)^2} 2\pi \int_0^\infty \mathrm{d}p\, p \frac{z}{\mathrm{e}^{\beta p^2/(2m)} - z} \\ &= \frac{A}{(2\pi\hbar)^2} \frac{2\pi m}{\beta} \int_0^\infty \mathrm{d}p\, \frac{\partial}{\partial p} \log\Bigl(1 - z\mathrm{e}^{-\beta p^2/(2m)}\Bigr) \\ &= \frac{A}{\lambda_\beta^2} \log \frac{1}{1 - z} = \frac{A}{\lambda_\beta^2} g_1(z)\,, \end{aligned}$$

where $\lambda_\beta = \hbar\sqrt{2\pi\beta/m}$ is the thermal wavelength and the fugacity z is from the range $0 < z < 1$. Since $g_1(z) = -\log(1 - z) \to \infty$ as $z \to 1$, we can have any number of bosons in the excited states and, therefore, there will not be an excess of bosons in the ground state. It follows that there is no Bose-Einstein condensation here.

3

(a) Since ρ_β is the extremal ρ, there is no contribution from the β-derivative of ρ_β, so that

$$S(\beta) = -\frac{\partial F(\beta)}{\partial T} = k_\mathrm{B}\beta^2 \frac{\partial F(\beta)}{\partial \beta} = -k_\mathrm{B}\mathrm{tr}\Bigl\{\rho_\beta \log \rho_\beta\Bigr\}\,.$$

(b) There is no change in ρ_β and $S(\beta)$, while $F(\beta) \to F(\beta) + E_0$.

(c) Since $\rho_\beta|E_k\rangle = |E_k\rangle\langle E_k|\rho_\beta|E_k\rangle = |E_k\rangle p_k$, we have $f(\rho_\beta)|E_k\rangle = |E_k\rangle f(p_k)$ for functions of ρ_β and, therefore,

$$S(\beta) = -k_\mathrm{B}\sum_k \langle E_k|\rho_\beta \log\rho_\beta|E_k\rangle = -k_\mathrm{B}\sum_k p_k\log p_k\,,$$

which is the Gibbs entropy formula if we regard the energy eigenstates as the microstates of the quantum system.

4

(a) For $J_2 = 0$, we have the standard Ising chain with N links of strength J_1 and the partition function as stated.

(b) For $J_1 = 0$, we have (i) the standard Ising chain with $\frac{1}{2}N$ links of strength J_2 together with (ii) a chain of $\frac{1}{2}N$ uncoupled sites. For (i) the partition function is $[2\cosh(K_2)]^{\frac{1}{2}N}$ and for (ii) is it $2^{\frac{1}{2}N}$. The product of these two contributions gives the partition function as stated.

(c) We sum over sites of the type "2" in the 123 triangle, each sum producing a factor

$$\sum_{s_2=\pm1} \mathrm{e}^{K_1(s_1s_2+s_2s_3)+K_2s_1s_3} = 2\cosh\Big((s_1+s_3)K_1\Big)\,\mathrm{e}^{K_2s_1s_3} = g\,\mathrm{e}^{\widetilde{K}s_1s_3}$$

with

$$g = 2\cosh(2K_1)^{\frac{1}{2}} \quad \text{and} \quad \mathrm{e}^{\widetilde{K}} = \cosh(2K_1)^{\frac{1}{2}}\mathrm{e}^{K_2}\,.$$

This reduces the zigzag chain with N next-neighbor and $\frac{1}{2}N$ next-next-neighbor links to a standard Ising chain with $\frac{1}{2}N$ next-neighbor links of strength $\widetilde{K}/\beta$ and no next-next-neighbor links, thereby picking up a factor of g for each of the $\frac{1}{2}N$ sites summed over. Accordingly,

$$\begin{aligned}Q(N,K_1,K_2) &= g^{\frac{1}{2}N}Q(\tfrac{1}{2}N,\widetilde{K},0)\\ &= 2^{\frac{1}{2}N}\cosh(2K_1)^{\frac{1}{4}N}\;2^{\frac{1}{2}N}\cosh(\widetilde{K})^{\frac{1}{2}N}\end{aligned}$$

or, with $2\cosh(\widetilde{K}) = \cosh(2K_1)^{\frac{1}{2}}\mathrm{e}^{K_2} + \cosh(2K_1)^{-\frac{1}{2}}\mathrm{e}^{-K_2}$,

$$\begin{aligned}Q(N,K_1,K_2) &= \Big[2\cosh(2K_1)\mathrm{e}^{K_2} + 2\mathrm{e}^{-K_2}\Big]^{\frac{1}{2}N}\\ &= 2^N\Big[\cosh(K_1)^2\cosh(K_2) + \sinh(K_1)^2\sinh(K_2)\Big]^{\frac{1}{2}N}\\ &= \Big[\mathrm{e}^{2K_1+K_2} + \mathrm{e}^{-2K_1+K_2} + 2\mathrm{e}^{-K_2}\Big]^{\frac{1}{2}N}\,.\end{aligned}$$

The particular cases of parts (a) and (b) are verified easily.

(d) For $K_1 = K_2 = K$ this gives, as stated,

$$Q(N,K,K) = \left(\mathrm{e}^{3K} + 3\mathrm{e}^{-K}\right)^{\frac{1}{2}N} = \mathrm{e}^{Nq(K)}$$
$$\text{with} \quad q(K) = \frac{1}{2}\log\left(\mathrm{e}^{3K} + 3\mathrm{e}^{-K}\right).$$

Then, the heat capacitance is

$$C = k_{\mathrm{B}} N K^2 \left(\frac{\partial}{\partial K}\right)^2 q(K) = 24 k_{\mathrm{B}} N \left(\frac{K}{\mathrm{e}^{2K} + 3\mathrm{e}^{-2K}}\right)^2 .$$
