1. **Thermal properties of a rubber band (20=8+4+8 marks)**
For a certain rubber band of length L that consists of n moles of rubber, the differential of the internal energy U is

$$\mathrm{d}U = T\,\mathrm{d}S + \tau\,\mathrm{d}L + \mu\,\mathrm{d}n\,,$$

where T, S, τ, μ are the temperature, entropy, tension, and chemical potential, respectively. The equation of state $TS = \alpha\tau L$ holds for this rubber band, where $\alpha > 0$ is a material-specific constant.

   (a) Show that
   $$U(S, L, n) = nf(S^\alpha L/n^{\alpha+1})$$
   with an undetermined function $f(\ )$.

   (b) What is the analog of the Gibbs-Duhem equation?

   (c) For $f(x) = \gamma x$ with $\gamma = \text{constant}$, find $\mu(T, \tau)$ and verify that the analog of the Gibbs-Duhem equation is obeyed.

2. **Two-dimensional boson gas (20 marks)**
An ideal gas of bosons is confined to an area of size A. Does Bose-Einstein condensation occur in such a two-dimensional boson gas? If yes, what is the critical temperature for given area A?

3. **Entropy of a quantum system (20=8+4+8 marks)**
A quantum system with Hamilton operator H has the free energy

$$F(\beta) = \underset{\rho}{\mathrm{Min}}\left(\mathrm{tr}\Big\{H\rho + \frac{1}{\beta}\rho\log\rho\Big\}\right) = \mathrm{tr}\Big\{H\rho_\beta + \frac{1}{\beta}\rho_\beta\log\rho_\beta\Big\}$$

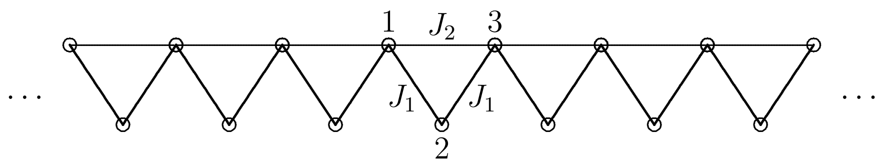
with $\rho_\beta = \mathrm{e}^{-\beta H}/\mathrm{tr}\big\{\mathrm{e}^{-\beta H}\big\}$, where the minimum is taken over all statistical operators ρ of this quantum system.

   (a) What is the entropy $S(\beta)$ in terms of ρ_β?

   (b) If a constant energy E_0 is added to H, that is: $H \to H + E_0$, what are the resulting changes in F, ρ_β, and S?

   (c) When $|E_k\rangle$ with $k = 1, 2, 3, \ldots$ are the pairwise orthogonal eigenkets of H, the probability of finding the system in the kth state is $p_k = \langle E_k|\rho_\beta|E_k\rangle$. Express $S(\beta)$ in terms of the p_ks and comment on the result.

4. A modified Ising model (40=4+8+16+12 marks)



A modified Ising model has N sites in a zigzag with links of strength J_1 between all pairs of next-neighbor sites and links of strength J_2 between every second pair of next-next-neighbor sites, so that the triangle 1,2,3 indicated in the picture contributes

$$-J_1(s_1 s_2 + s_2 s_3) - J_2 s_1 s_3 \quad \text{with } s_j = \pm 1$$

to the energy. For $K_1 = \beta J_1$ and $K_2 = \beta J_2$, we denote the canonical partition function by $Q(N, K_1, K_2)$.

(a) Show that $Q(N, K_1, 0) = 2^N \cosh(K_1)^N$ is the partition function for $J_2 = 0$.

(b) Show that $Q(N, 0, K_2) = 2^N \cosh(K_2)^{\frac{1}{2}N}$ is the partition function for $J_1 = 0$.

(c) Explain why a relation of the form

$$Q(N, K_1, K_2) = g^{\frac{1}{2}N} Q(\tfrac{1}{2}N, \widetilde{K}, 0)$$

must hold and determine g and $\widetilde{K}$ as functions of K_1 and K_2. Then find $Q(N, K_1, K_2)$.

(d) Verify that the partition function is

$$Q(N, K, K) = \left(\mathrm{e}^{3K} + 3\mathrm{e}^{-K}\right)^{\frac{1}{2}N}$$

when $K_1 = K_2 = K$. Find the heat capacitance in this case.