(a) Writing $F(T, V) = -\dfrac{\pi^2}{45}\dfrac{V}{(\hbar c)^3}(k_{\mathrm{B}}T)^4$, we have $S = -\left(\dfrac{\partial F}{\partial T}\right)_V = -\dfrac{4}{T}F$ and

$(k_{\mathrm{B}}T)^3 = \dfrac{45}{4\pi^2}\dfrac{S}{k_{\mathrm{B}}}\dfrac{(\hbar c)^3}{V}$. This gives

$$U(S, V) = F + TS = -3F = \dfrac{\pi^2}{15}\dfrac{V}{(\hbar c)^3}\left(\dfrac{45}{4\pi^2}\dfrac{S}{k_{\mathrm{B}}}\dfrac{(\hbar c)^3}{V}\right)^{4/3}$$

$$= \dfrac{3}{4}\left(\dfrac{45}{4\pi^2}\right)^{1/3}\left(\dfrac{S}{k_{\mathrm{B}}}\right)^{4/3}\left(\dfrac{V}{(\hbar c)^3}\right)^{-1/3}.$$

Then $P = -\left(\dfrac{\partial U}{\partial V}\right)_T = \dfrac{1}{3V}U$ gives $\dfrac{V}{(\hbar c)^3} = \dfrac{1}{4}\left(\dfrac{45}{\pi^2}\right)^{1/4}\left((\hbar c)^3 P\right)^{-3/4}\dfrac{S}{k_{\mathrm{B}}}$ and

$$H(S, P) = U + PV = \dfrac{4}{3}U = \left(\dfrac{45}{\pi^2}\right)^{1/4}\dfrac{S}{k_{\mathrm{B}}}\left((\hbar c)^3 P\right)^{1/4}.$$

Finally,

$$G = F + PV = F - V\left(\dfrac{\partial F}{\partial V}\right)_T = 0 \quad \text{or} \quad G = H - TS = H - S\left(\dfrac{\partial H}{\partial S}\right)_P = 0$$

since $\left(\dfrac{\partial F}{\partial V}\right)_T = \dfrac{1}{V}F$ and $\left(\dfrac{\partial H}{\partial S}\right)_P = \dfrac{1}{S}H$.

(b) Since $P = -\left(\dfrac{\partial F}{\partial V}\right)_T \propto T^4$ does not depend on V, the isothermal curves are simply those with constant P. And since $S \propto VT^3 \propto VP^{3/4}$, the isentropic curves are those with $V^4 P^3 = \text{constant}$.

(c) In $C_P = T\left(\dfrac{\partial S}{\partial T}\right)_P$, we need changes of the temperature for constant pressure but, for the photon gas, P is a function of T only, and so T is constant when P is constant. Therefore, there is no meaningful C_P value.

(a) The single-particle energies are $\varepsilon_j = j\hbar\omega$ with $j = 0$ for the ground state and $j = 1, 2, 3, \ldots$ for the excited states. Accordingly, the expected number of bosons in the ground state and in the exited states are

$$\langle N_0 \rangle = \dfrac{z}{1 - z}, \quad \langle N_{\mathrm{ex}} \rangle = \sum_{j=1}^{\infty}\dfrac{z}{e^{\beta\varepsilon_j} - z} = \sum_{j=1}^{\infty}\dfrac{z}{e^{j\beta\hbar\omega} - z}.$$

(b) For low temperature, $\beta\hbar\omega \gg 1$, we have $\dfrac{z}{\mathrm{e}^{j\beta\hbar\omega} - z} \cong z\mathrm{e}^{-j\beta\hbar\omega}$ and only the $j = 1$ term matters in the sum, so that

$$\langle N_{\mathrm{ex}}\rangle \cong z\mathrm{e}^{-\beta\hbar\omega} = \frac{\langle N_0\rangle}{\langle N_0\rangle + 1}\mathrm{e}^{-\beta\hbar\omega}\,.$$

(c) The sum over j for $\langle N_{\mathrm{ex}}\rangle$ in (a) converges to a finite value for all positive temperatures and all values of the fugacity $z \le 1$, including $z = 1$. It follows that there is a maximum number of bosons than can be in the excited states for $T > 0$.

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3 We recall that an Ising chain with N particles, no on-site energy, and $N - 1$ next-neighbor interaction links of strength J has a free energy of

$$F = -k_{\mathrm{B}}T \log\Big(2[2\cosh(K)]^{N-1}\Big),$$

which is $F = -Nk_{\mathrm{B}}T \log\big(2\cosh(K)\big)$ for $N \gg 1$, with $K = \beta J = J/(k_{\mathrm{B}}T)$.

(a) For $J' = 0$, we just have two Ising chains with $\frac{1}{2}N \gg 1$ particles each, so that

$$F(T, J, 0) = 2 \times \left[-\frac{1}{2}Nk_{\mathrm{B}}T \log\big(2\cosh(K)\big)\right] = -Nk_{\mathrm{B}}T \log\big(2\cosh(K)\big)\,.$$

(b) For $J = 0$, we have $\frac{1}{2}N$ two-particle chains with one link each, so that

$$F(T, 0, J') = \frac{1}{2}N \times \left[-k_{\mathrm{B}}T \log\big(4\cosh(K')\big)\right] = -\frac{1}{2}Nk_{\mathrm{B}}T \log\big(4\cosh(K')\big),$$

with $K' = \beta J' = J'/(k_{\mathrm{B}}T)$.

(c) We have $F(T, J, J') = F(T, J, 0) + \big\langle E_k^{(1)}\big\rangle^{(0)}$ with $^{(0)}$ referring to the $J' = 0$ situation of (a), and

$$E_k^{(1)} = -J' \sum_{j=1}^{\frac{1}{2}N} s_{1j}s_{2j},$$

where s_{1j} is the s value for site j in chain 1 and s_{2j} is the s value for the partner site in chain 2. Since these values are uncorrelated and $\langle s_j\rangle = 0$ for an Ising chain without on-site energy, we have $\langle s_{1j}s_{2j}\rangle^{(0)} = \langle s_{1j}\rangle^{(0)}\langle s_{2j}\rangle^{(0)} = 0$ and so get

$$F(T, J, J') = -Nk_{\mathrm{B}}T \log\big(\cosh(K)\big) + \mathcal{O}(J'^2)\,.$$

There is no first-order term.

(a) In (4.1.9), we do not have the contribution from the electron-electron interaction and write $\dfrac{-Ze^2}{r_0}$ for the chemical potential μ. This gives

$$\rho(r) = \frac{1}{3\pi^2}\left[\frac{2m}{\hbar^2}\left(\frac{Ze^2}{r} - \frac{Ze^2}{r_0}\right)\right]_+^{3/2} = \frac{1}{3\pi^2}\left[\frac{2Z}{a_0 r_0}\left(\frac{r_0}{r} - 1\right)\right]_+^{3/2}$$

with $\dfrac{\hbar^2}{m} = e^2 a_0$. Then, (4.6.1) reads

$$Z = \frac{4\pi}{3\pi^2}\left(\frac{2Z}{a_0 r_0}\right)^{3/2}\int_0^{r_0} dr\, r^2\left(\frac{r_0}{r} - 1\right)^{3/2}$$

$$= \frac{4}{3\pi}\left(2Z\frac{r_0}{a_0}\right)^{\frac{3}{2}}\underbrace{\int_0^1 dx\, x^2\left(\frac{1}{x} - 1\right)^{\frac{3}{2}}}_{= \frac{(\frac{1}{2})!\,(\frac{3}{2})!}{3!} = \frac{1}{12}\frac{3\pi}{4}} = \frac{1}{12}\left(2Z\frac{r_0}{a_0}\right)^{\frac{3}{2}}$$

and thus gives $r_0 = (18)^{1/3}Z^{-1/3}a_0$.

(b) For the kinetic energy we get

$$E_{\text{kin}} = \int (d\boldsymbol{r})\frac{\hbar^2}{10\pi^2 m}[3\pi^2\rho(r)]^{5/3}$$

$$= \frac{4\pi}{10\pi^2}e^2 a_0\left(\frac{2Z}{a_0 r_0}\right)^{\frac{5}{2}}r_0^3\underbrace{\int_0^1 dx\, x^2\left(\frac{1}{x} - 1\right)^{\frac{5}{2}}}_{= \frac{(-\frac{1}{2})!\,(\frac{5}{2})!}{3!} = \frac{1}{8}\frac{5\pi}{2}}$$

$$= \frac{1}{8}\frac{2Ze^2}{r_0}\underbrace{\left(2Z\frac{r_0}{a_0}\right)^{\frac{3}{2}}}_{= 12Z} = 3Z^2\frac{e^2}{r_0} = \left(\frac{3}{2}\right)^{1/3}Z^{7/3}\frac{e^2}{a_0}\,,$$

and the potential energy is

$$E_{\text{pot}} = -\int (d\boldsymbol{r})\frac{Ze^2}{r}\rho(r) = -\frac{4}{3\pi}Ze^2 r_0^2\left(\frac{2Z}{a_0 r_0}\right)^{\frac{3}{2}}\underbrace{\int_0^1 dx\, x\left(\frac{1}{x} - 1\right)^{\frac{3}{2}}}_{= \frac{(-\frac{1}{2})!\,(\frac{3}{2})!}{2!} = \frac{1}{2}\frac{3\pi}{4}}$$

$$= -\frac{1}{2}\frac{Ze^2}{r_0}12Z = -6Z^2\frac{e^2}{r_0} = -2E_{\text{kin}}\,,$$

so that

$$E = E_{\text{kin}} + E_{\text{pot}} = -E_{\text{kin}} = -\left(\frac{3}{2}\right)^{1/3}Z^{7/3}\frac{e^2}{a_0}\,.$$

(c) We have $\boxed{?} = (3/2)^{1/3} > 1 > 0.7687$ because we ignore the repulsive electron-electron interaction and, as a consequence, the binding energy is larger.