Problem 1 (**25**=10+10+5 marks)
We know that

$$F(T,V) = -\frac{\pi^2}{45}\frac{k_\mathrm{B}^4}{(\hbar c)^3} V T^4$$

is the free energy of the photon gas.

(a) Find the internal energy $U(S,V)$, the enthalpy $H(S,P)$, and the free enthalpy $G(T,P)$ as functions of their natural variables.

(b) In a V,P diagram, which curves identify isothermal processes? Which curves identify isentropic processes?

(c) Explain why there is no meaningful value of C_P, the heat capacity for constant pressure, for a photon gas?

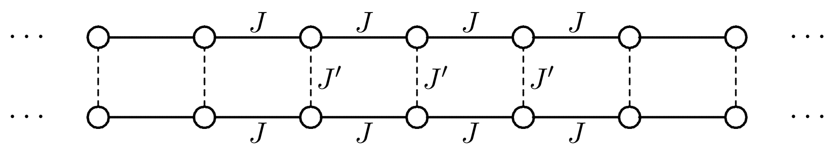
Problem 2 (**25**=10+5+10 marks)
Consider a one-dimensional ideal quantum gas of bosons in a harmonic-oscillator trap with energy spacing $\hbar\omega$.

(a) For fugacity z and temperature $T = (k_\mathrm{B}\beta)^{-1}$, what is the expected number of bosons in the excited states? [Just write down an equation "$\langle N_\mathrm{ex}\rangle = \cdots$"; you do not need to evaluate the expression.]

(b) What approximates $\langle N_\mathrm{ex}\rangle$ for low temperatures?

(c) For temperature $T > 0$, can you have any number of bosons in the excited states, or is there a maximum number? Justify your answer.

Problem 3 (**20**=5+5+10 marks)



We have two long parallel Ising chains, each made up of $\frac{1}{2}N$ particles with no on-site energy and a next-neighbor interaction of strength J (solid connecting lines). There is also an interaction of strength J' between each particle of one chain and the nearest particle of the other chain (dashed connecting lines).

(a) For $J' = 0$, what is the free energy $F(T,J,J'=0)$?

(b) For $J = 0$, what is the free energy $F(T,J=0,J')$?

(c) For $0 < J' \ll J$, what is the free energy $F(T,J,J')$ to first order in J'?

Problem 4 (**30**=15+10+5 marks)
As a very rough approximation of a neutral atom with Z electrons, consider the Thomas–Fermi model without the repulsive electron-electron interaction.

(a) Show that the electron density is of the form

$$\rho(r) = \frac{1}{3\pi^2}\left[\frac{2m}{\hbar^2}\left(\frac{Ze^2}{r} - \frac{Ze^2}{r_0}\right)\right]_+^{3/2}$$

with $r_0 > 0$, and determine the value of r_0 as a function of Z.

(b) Then confirm that the energy is proportional to $Z^{7/3}$ and determine the proportionality factor $\boxed{?}$ in $E(Z) = -\boxed{?}\,Z^{7/3}\frac{e^2}{a_0}$?

(c) Is $\boxed{?}$ larger or smaller than the value 0.7687 of the standard Thomas–Fermi model? Explain why.

Hint: Euler's beta-function integral $\int_0^1 \mathrm{d}x\, x^\alpha (1-x)^\beta = \dfrac{\alpha!\,\beta!}{(\alpha+\beta+1)!}$ could be useful.