1. **Carnot cycle in a photon gas (15=10+5 marks)**
Consider a Carnot cycle $(T_1, V_1) \to (T_2, V_2) \to (T_3, V_3) \to (T_4, V_4) \to (T_1, V_1)$ with $T_1 = T_2 > T_3 = T_4$ and $S_4 = S_1 < S_2 = S_3$ (entropy) for a photon gas with its well-known free energy

$$F(T, V) = -\frac{\pi^2}{45}\frac{k_{\mathrm{B}}^4}{(\hbar c)^3} V T^4 \,.$$

(a) How much heat is absorbed by the photon gas, and how much work does the photon gas do on its environment, in the four transitions from (T_j, V_j) to $(T_{j'}, V_{j'})$?

(b) Determine the efficiency of this engine (= the total work done, divided by the heat absorbed from the hot reservoir). Do you get the expected value?

Hint: It is expedient to express all quantities in terms of T_1, T_3, S_1, and S_3.

2. **Clausius gas near the critical point (30=15+15 marks)**
The so-called Clausius model of a real gas is specified by the equation of state

$$P(T, v) = \frac{RT}{v - b} - \frac{a}{(v + c)^2 T}$$

with positive material constants a, b, and c. Just like the van der Waals gas, the Clausius gas has a gas-to-liquid phase transition for temperatures below the critical temperature T_{cr}.

(a) Express the critical temperature T_{cr} and also the critical values of the molar volume (v_{cr}) and the pressure (P_{cr}) in terms of a, b, c, and the gas constant R. What is the value of $\dfrac{P_{\mathrm{cr}} v_{\mathrm{cr}}}{RT_{\mathrm{cr}}}$?

(b) Find the coexistence pressure $P(T)$ for temperatures just below the critical temperature, $0 \lesssim T_{\mathrm{cr}} - T \ll T_{\mathrm{cr}}$.

3. Two-dimensional quantum gas of bosons (25=10+5+10 marks)
Consider a two-dimensional ideal quantum gas of bosons in a harmonic-oscillator trap with energy spacing $\hbar\omega$.

(a) For fugacity z and temperature $T = (k_B\beta)^{-1}$, what is the expected number of bosons in the excited states? [Just write down an equation " $\langle N_{\mathrm{ex}}\rangle = \cdots$"; you do not need to evaluate the expression.]

(b) What approximates $\langle N_{\mathrm{ex}}\rangle$ for low temperatures?

(c) Is there Bose–Einstein condensation in this two-dimensional system? Justify your answer.

4. Classical gas with potential energy (30=5+10+15 marks)
Consider an ideal classical gas of N particles with the energy

$$H(\boldsymbol{r}_1, \boldsymbol{p}_1; \boldsymbol{r}_2, \boldsymbol{p}_2; \ldots; \boldsymbol{r}_N, \boldsymbol{p}_N) = \sum_{j=1}^{N} \left[\frac{\boldsymbol{p}_j^2}{2m} + w r_j^3\right] ,$$

where $r_j = |\boldsymbol{r}_j|$ is the length of the position vector of the jth particle and $w > 0$ is a constant.

(a) Which of the system parameters m, w, and N are extensive variables, which are intensive variables? Why?

(b) Find the canonical partition function.

(c) Then determine the average energy per particle in units of $k_B T$ as well as the average kinetic energy per particle and the average potential energy per particle. Verify that the sum of average kinetic energy and average potential energy equals the average energy.