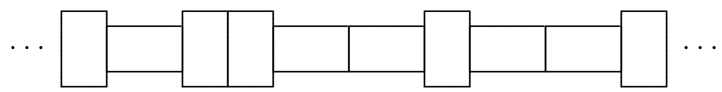
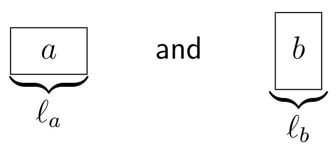
1. **Warming up (10 marks)**
   You add a milli-calorie of heat ($\simeq 4 \times 10^{-3}\,\mathrm{J}$) to a certain substance at a temperature of $300\,\mathrm{K}$ (water, perhaps, but that is not relevant). What is the corresponding change in the number of microstates available to the system?

   Hint: Just get an estimate by a back-of-the-envelope calculation.

2. **A simple rubber-band model (30=8+6+8+8 marks)**
   A physical model of a rubber band is a single long chain in which $N$ molecules are linked:



   Each molecule has two possible configurations:




   which have lengths $\ell_a, \ell_b$ and energies $E_a = 0$ and $E_b = E_0 > 0$. The rubber band is in thermal contact with the atmosphere at temperature $T = 1/(k_B\beta)$.

   **(a)** Determine the canonical partition function $Q(\beta, N)$ for this system and then find the entropy $S(\beta, N)$.

   **(b)** What is the average number of molecules with energy $E_a$?

   **(c)** What is $\Omega(E, N)$, the number of microstates with energy $E$? Is the entropy obtained as $S = k_B \log \Omega$ equal to the entropy found in part (a)?

   **(d)** Find the average length $L(T, N)$ of the rubber band. When the temperature is increased, does $L(T, N)$ increase or decrease? What is $L(T, N)$ for very low temperatures ($k_B T \ll E_0$) and for very high temperatures ($k_B T \gg E_0$)?

**3. A dilute gas (30=5+7+14+4 marks)**

The grand-canonical partition function $Z(\beta, V, z)$ of a dilute gas is given by $\log Z = (k_{\mathrm{B}} T_0 \beta)^{-\kappa} \dfrac{V}{V_0} z$, where $T_0$, $V_0$, and $\kappa$ are positive material constants.

**(a)** Confirm that $PV = \mathrm{constant}$ for isothermal changes.

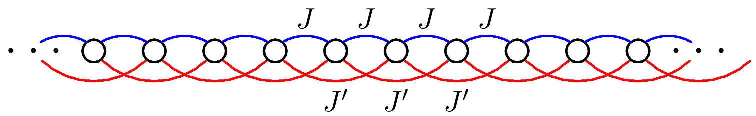
**(b)** What is the equation of state for isentropic changes? Confirm that you get the expected answer when $\kappa = \dfrac{3}{2}$.

**(c)** Determine the free energy $F(T, V, N)$ and the internal energy $U(S, V, N)$.

**(d)** What are the heat capacities for constant volume and constant pressure?

**4. Ising with next-next-neighbor interaction (30=4+4+12+10 marks)**

Consider a modified Ising chain (or ring) with $N$ sites, no on-site energy, and next-neighbor interaction strength $J$. There is also a next-next-neighbor interaction of strength $J'$. Symbolically, then, we have this picture:




As usual, we use $K = \beta J$ and $K' = \beta J'$ for convenience.

**(a)** What is the free energy $F(K, 0, N)$ when $K' = 0$?

**(b)** What is the free energy $F(0, K', N)$ when $K = 0$?

**(c)** For $K \neq 0$ and $K' \neq 0$, find the canonical partition function and then find the free energy $F(K, K', N)$. Verify that you get the expected expressions when $K' = 0$ or $K = 0$.

**(d)** Determine the heat capacity to the leading order in $K'$ when $0 < K' \ll 1$.

Hint: Remember that $s_{j-1}s_{j+1} = (s_{j-1}s_j)\,(s_j s_{j+1})$.

End of Paper

(BG Englert)