Tutorial for Chapter 5

1. Using (0.42), prove Parseval's theorem (0.43).

2. Memorize the concepts of fringe visibility and coherence time

3. Show that the degree of coherence function $\mathrm{Re}\{\gamma(\tau)\}$ reduces to $\cos(\omega_0\tau)$ in the case of a plane wave $E(t) = E_0 e^{i(k_0 z - \omega_0 t)}$ being sent through a Michelson interferometer. In other words, the output intensity from the interferometer reduces to

$$I = 2I_0\,[1 + \cos(\omega_0\tau)]$$

4. Light emerging from a dense hot gas has a collisionally broadened power spectrum described by the Lorentzian function

$$I(\omega) = \frac{I(\omega_0)}{1 + \left(\frac{\omega - \omega_0}{\Delta\omega_{\mathrm{FWHM}}/2}\right)^2}$$

The light is sent into a Michelson interferometer. What is the coherence time τ_c of the light? HINT: See (0.46).

5. Verify (5.19). HINT: Write $\gamma = |\gamma|\, e^{i\phi}$ and assume that the oscillations in that give rise to fringes are due entirely to changes in ϕ and that $|\gamma|$ is a slowly varying function in comparison to the oscillations.

6. Show that the fringe visibility (5.21) of Gaussian distribution (5.20) goes from 1 to $e^{-\pi/2} = 0.21$ as the round-trip path in one arm of the instrument is extended by a coherence length.

7. Find the FWHM bandwidth in wavelength $\Delta\lambda_{\mathrm{FWHM}}$ in terms of the coherence length ℓ_c and the center wavelength λ_0 associated with (5.20).
HINT: Derive $\Delta\omega_{\mathrm{FWHM}} = 2\sqrt{\ln 2}\,\Gamma$. To convert to a wavelength difference, use $\lambda = \frac{2\pi c}{\omega} \Rightarrow \Delta\lambda \cong -\frac{2\pi c}{\omega^2}\Delta\omega$. You can ignore the minus sign; it simply means that wavelength decreases as frequency increases.

8. A point source with wavelength $\lambda = 500$ nm illuminates two parallel slits separated by $h = 1.0$ mm. If the screen is $D = 2$ m away, what is the separation between the two neighboring bright fringes on the screen?

9. A thin piece of glass with thickness $d = 0.01$ mm and index $n = 1.5$ is placed in front of one of the slits. By how many fringes does the pattern at the screen move? HINT: This effectively introduces a relative phase $\Delta\phi$ in (5.32). Compare the phase of the light when traversing the glass versus traversing an empty region of the same thickness.

10. Design an experiment to carefully measure the separation of a double slit in the lab ($h \approx 1$ mm separation) by shining a He-Ne laser ($\lambda = 633$ nm) through it and measuring the bright fringe separations on a distant wall (say, $D = 2.00$ m). HINT: You can think of the very temporally and spatially coherent laser light as coming from a distant source. For better accuracy, measure across several fringes and divide.

11. A light pulse has a temporal profile defined in (5.29) with $\lambda_0 = 780$ nm and $T = 20 \times 10^{-15}$ s. Find $\Delta\lambda_{FWHM}$ of the light pulse.

12. A square light pulse is given by

$$E(t) = \begin{cases} E_0 e^{i(k_0 z - \omega_0 t)} & |t| \leq T/2 \\ 0 & |t| > T/2 \end{cases}$$

where E_0 is a constant. Find its frequency spectrum by using the Fourier transform.

1.

$$\int I(r,t)\,dt$$

$$= \frac{1}{2}\varepsilon_0 n c \int_{-\infty}^{\infty} E(r,t)\cdot E^*(r,t)\,dt$$

$$= \frac{1}{2}\varepsilon_0 n c \int_{-\infty}^{\infty} dt \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(r,\omega) e^{-i\omega t} d\omega \left(\int_{-\infty}^{\infty} \frac{E(r,\omega') e^{-i\omega' t} d\omega'}{\sqrt{2\pi}} \right)^*$$

$$= \frac{1}{2}\varepsilon_0 n c \int_{-\infty}^{\infty} dt \frac{1}{2\pi} \int_{-\infty}^{\infty} E(r,\omega) e^{-i\omega t} d\omega \left(\int_{-\infty}^{\infty} E^*(r,\omega') e^{i\omega' t} d\omega' \right)$$

$$= \frac{1}{2}\varepsilon_0 n c \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' E(r,\omega) E^*(r,\omega') \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega'-\omega)t} dt$$

$$= \frac{1}{2}\varepsilon_0 n c \int_{-\infty}^{\infty} d\omega\, E(r,\omega) \int_{-\infty}^{\infty} d\omega' E^*(r,\omega') \delta(\omega'-\omega)$$

$$= \frac{1}{2}\varepsilon_0 n c \int_{-\infty}^{\infty} d\omega\, E(r,\omega) E^*(r,\omega)$$

$$= \int_{-\infty}^{\infty} I(r,\omega)\, d\omega$$

3. $$\gamma(\tau) = \frac{\int_{-\infty}^{\infty} I(\omega)\, e^{-i\omega\tau}\, d\omega}{\int_{-\infty}^{\infty} I(\omega)\, d\omega}$$

$$E(t) = E_0\, e^{i(k_0 z - \omega_0 t)}$$

<u>F.T.</u>

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_0\, e^{i(k_0 z - \omega_0 t)}\, e^{i\omega t}\, dt$$

$$= \frac{E_0\, e^{ik_0 z}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(\omega - \omega_0)t}\, dt$$

$$= \sqrt{2\pi}\, E_0\, e^{ik_0 z}\, \delta(\omega - \omega_0)$$

$$E^*(\omega) = \sqrt{2\pi}\, E_0\, e^{-ik_0 z}\, \delta(\omega - \omega_0)$$

$$\int_{-\infty}^{\infty} I(\omega)\, d\omega = 2\pi E_0^2 \int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, \delta(\omega - \omega_0)\, d\omega$$

$$= 2\pi E_0^2 \,[\, 1 \,] = 2\pi E_0^2$$

$$\int_{-\infty}^{\infty} 2\pi E_0^2\, \delta(\omega - \omega_0)\, \delta(\omega - \omega_0)\, e^{-i\omega\tau}\, d\omega$$

$$= 2\pi E_0^2\, e^{-i\omega_0 \tau}$$

$$\gamma(\tau) = e^{-i\omega_0 \tau}.$$

$$\text{Re}\, \gamma(\tau) = \cos(\omega_0 \tau)$$

$$I = 2 I_0 \,[\, 1 + \cos(\omega_0 \tau) \,]$$

4.

$$\gamma(\tau) = \frac{\int_{-\infty}^{\infty} I(\omega) e^{-i\omega\tau} d\omega}{\int_{-\infty}^{\infty} I(\omega)\, d\omega}$$

$\Delta\omega = \frac{\Delta\omega_{FWHM}}{2}$

$\omega - \omega_0 = u$

$$= \frac{\int_{-\infty}^{\infty} \frac{I(\omega_0)}{1+\left(\frac{\omega-\omega_0}{\Delta\omega}\right)^2} e^{-i\omega\tau} d\omega}{\int_{-\infty}^{\infty} \frac{I(\omega_0)}{1+\frac{(\omega-\omega_0)^2}{\Delta\omega^2}} d\omega}$$

$$= \frac{e^{-i\omega_0\tau}\int_{-\infty}^{\infty} e^{-iu\tau} / (1+u^2/\Delta\omega^2)\, du}{\int_{-\infty}^{\infty} \frac{1}{1+u^2/\Delta\omega^2} d\omega}$$

$$= e^{-i\omega_0\tau} \frac{\frac{\pi(\Delta\omega)}{2} e^{-\Delta\omega\tau}}{\frac{\pi(\Delta\omega)}{2}}$$

$$= e^{-i\omega_0\tau} e^{-\Delta\omega\tau}$$

$$\tau_c \equiv 2\int_0^{\infty} |\gamma(\tau)|^2 d\tau = 2\int_0^{\infty} e^{-2\Delta\omega\tau} d\tau$$

$$= \frac{1}{\Delta\omega} = \frac{2}{\Delta\omega_{FWHM}}$$

5.

$$I_{det} = I_0 [1 + \mathrm{Re}\,\gamma(\tau)]$$

$$= I_0 [1 + |\gamma(\tau)| \mathrm{Re}\, e^{i\phi}]$$

$$= I_0 [1 + |\gamma(\tau)| \cos\phi .$$

$$I_{det}^{max} = I_0 [1 + |\gamma(\tau)|]$$

$$I_{det}^{min} = I_0 [1 - |\gamma(\tau)|]$$

$$V(\tau) = \frac{I_{det}^{max} - I_{det}^{min}}{I_{det}^{max} + I_{det}^{min}} = \frac{2I_0 |\gamma(\tau)|}{2I_0}$$

6. Visibility $= |\gamma(\tau)| = \left| e^{-i\tau\omega_0 - \frac{W^2\tau^2}{4}} \right|$

$$= e^{-\frac{W^2\tau^2}{4}} \Rightarrow e^{-\frac{\pi}{2}} \Big|_{\tau = \tau_{\frac{\pi}{2}}}$$

$$\frac{W^2 \tau_{\frac{\pi}{2}}^2}{4} = \frac{\pi}{2} \qquad \tau_{\frac{\pi}{2}} = \frac{\sqrt{2\pi}}{W}$$

Coherent time $\tau_c = 2\int_0^\infty |\gamma(\tau)|^2 d\tau$

$$= 2\int_0^\infty e^{-\frac{2W^2\tau^2}{4}} d\tau = \frac{2\sqrt{2}}{W}\int_0^\infty e^{-\frac{W^2\tau^2}{2}} d\left(\frac{W\tau}{\sqrt{2}}\right)$$

$$= \frac{2\sqrt{2}}{W}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{W} = \tau_{\pi/2}$$

Coherent length $= \frac{\sqrt{2\pi}\, c}{W}$

7. $|\Delta\lambda_{FWHM}| = \frac{2\pi c}{\omega_0^2} \Delta\omega_{FWHM}$

$$\frac{|I(\omega)|}{I(\omega_0)} = e^{-\frac{(\omega-\omega_0)^2}{W^2}} = \frac{1}{2} \quad \text{when } \omega - \omega_0 = \frac{\Delta\omega_{FWHM}}{2}$$

$$-\frac{\left(\frac{\Delta\omega_{FWHM}}{2}\right)^2}{W^2} = -\ln 2$$

$$\Delta\omega_{FWHM} = 2\sqrt{\ln 2}\, W = 2\sqrt{\ln 2}\,\frac{W}{\sqrt{2\pi}\, c}\, c\sqrt{2\pi}$$

$$= \frac{2c\sqrt{2\pi \ln 2}}{l_c}$$

$$|\Delta\lambda_{FWHM}| = \frac{2\pi c}{\left(\frac{2\pi c}{\lambda_0}\right)^2}\frac{\sqrt{2\pi\ln 2}\, 2c}{l_c} = \frac{2\lambda_0^2}{l_c}\sqrt{\frac{\ln 2}{2\pi}}$$

8.

$$\Delta\phi = 0$$

$$\frac{khy}{D} = 2\pi m \qquad m = 0, 1, \cdots$$

$$\frac{kh\Delta y}{D} = 2\pi \Delta m \qquad \Delta m = 1$$

$$kh\Delta y = 2\pi D$$

$$\Delta y = \frac{2\pi D}{\frac{2\pi}{\lambda} h} = \frac{\lambda D}{h}$$

$$= \frac{500 \times 10^{-9} m \times 2 m}{1 \times 10^{-3} m}$$

$$= 10^{3} \times 10^{-3} \times 10^{-9} m = 10^{-3} m$$

$$= 1 mm$$

9.

$$\frac{khy}{D} + \Delta\phi = \frac{khy}{D} + k(n-1)d$$

$$k(n-1)d = 2\pi m$$

$$\frac{2\pi}{\lambda}(n-1)d = 2\pi m$$

$$m = \frac{(n-1)d}{\lambda} = \frac{(1.5-1)\times 0.01\times 10^6\, nm}{500\, nm}$$

$$= \frac{0.5\times 0.01\times 10^6}{5\times 10^2} = \frac{5\times 10^{-3}\times 10^{-2}\times 10^6}{5}$$

$$= 10$$

10 fringes more !

10. $$I = 2I_0\left[1+\cos\left(\frac{2\pi}{\lambda}\frac{hy}{D}\right)\right]$$

On axis $y=0$ $I = 4I_0$

Displaced from the axis $\frac{2\pi}{\lambda}\frac{hy}{D} = 2\pi\times m$

Count on $m=5$, $\frac{2\pi}{\lambda}\frac{hy}{D} = 2\pi\times 5$

$$\Rightarrow h = \frac{5\lambda D}{y} = \frac{5\times 633\, nm\times 2m}{y}$$

y ← measured !

11. $W = \frac{\sqrt{2}}{T}$ or $W = \frac{1}{T}$

$$\Delta\omega_{FWHM} = 2\sqrt{\ln 2}\, W = \frac{2\sqrt{\ln 2}}{T}$$

$$|\Delta\lambda_{FWHM}| = \frac{2\pi c}{\omega_0^2}\,\Delta\omega_{FWHM}$$

$$= \frac{2\pi c}{\left(\frac{2\pi c}{\lambda_0}\right)^2}\,\frac{2\sqrt{\ln 2}}{T} = \frac{\lambda_0^2}{2\pi c}\,\frac{2\sqrt{\ln 2}}{T}$$

$$= \frac{\lambda_0^2}{T}\,\frac{\sqrt{\ln 2}}{\pi c} = \frac{(780)^2}{20\times10^{-15}}\,\frac{\sqrt{\ln 2}}{\pi\times3\times10^8\times10^9}$$

$$= 26\ nm$$

or 38 nm

12.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-T/1}^{T/2} E_0\, e^{i(k_0 z - \omega_0 t)}\, e^{i\omega t}\, dt$$

$$= \frac{E_0 e^{ik_0 z}}{\sqrt{2\pi}\,(\omega-\omega_0) i}\int_{T/2}^{T/2} e^{i(\omega-\omega_0)t}\, d\,i(\omega-\omega_0)t$$

$$= \frac{E_0 e^{ik_0 z}}{\sqrt{2\pi}\,(\omega-\omega_0) i}\, e^{i(\omega-\omega_0)t}\Big|_{-T/2}^{T/2}$$

$$= \frac{E_0 e^{ik_0 z}}{\sqrt{2\pi}}\, T\, \frac{e^{i(\omega-\omega_0)T/2} - e^{-i(\omega-\omega_0)T/2}}{2i(\omega-\omega_0)\,T/2}$$

$$= \frac{E_0 e^{ik_0 z}\, T}{\sqrt{2\pi}}\left[\frac{\sin(\omega-\omega_0)T/2}{(\omega-\omega_0)T/2}\right]$$