First-order ordinary differential equations

Fist-degree first-order equations

First-degree first-order ODEs contain only dy/dx equated to some function of x and y, and can be written in either of two equivalent standard forms

$$\frac{dy}{dx} = F(x, y),$$

or

$$A(x, y)\, dx + B(x, y)\, dy = 0,$$

where $F(xmy) = -A(x, y)/B(x, y)$, and $F(x, y)$, $A(x, y)$ and $B(x, y)$ are in general functions of both x and y.

Separable-variable equations

A separable-variable equation is one which may be written in the conventional form

$$\frac{dy}{dx} = f(x)g(y), \tag{1}$$

where $f(x)$ and $g(y)$ are functions of x and y respectively. Rearranging this equation, we obtain

$$\int \frac{dy}{g(y)} = \int f(x)\,dx.$$

Finding the solution $y(x)$ that satisfies Eq. (1) then depends only on the ease with which the integrals in the above equation can be evaluated.

Example

Solve

$$\frac{dy}{dx} = x + xy.$$

Answer

Since the RHS of this equation can be factorized to give $x(1+y)$, the equation becomes separable and we obtain

$$\int \frac{dy}{1+y} = \int x\,dx$$

Now integrating both sides, we find

$$\ln(1+y) = \frac{x^2}{2} + c,$$

and so

$$1 + y = \exp\left(\frac{x^2}{2} + c\right) = A\exp\left(\frac{x^2}{2}\right),$$

where c and hence A is an arbitrary constant.

Exact equation

An exact first-degree first-order ODE is one of the form

$$A(x,y)\,dx+B(x,y)\,dy=0 \text{ and for which } \frac{\partial A}{\partial y}=\frac{\partial B}{\partial x}. \tag{2}$$

In this case, $A(x,y)\,dx+B(x,y)\,dy$ is an exact differential, $dU(x,y)$ say. That is,

$$A\,dx+B\,dy=dU=\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy,$$

from which we obtain

$$A(x,y)=\frac{\partial U}{\partial x}, \tag{3}$$

$$B(x,y)=\frac{\partial U}{\partial y}. \tag{4}$$

Since $\partial^2 U/\partial x \partial y = \partial^2 U/\partial y \partial x$, we therefore require

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \tag{5}$$

If Eq. (5) holds then Eq. (2) can be written $dU(x, y) = 0$, which has the solution $U(x, y) = c$, where c is a constant and from Eq. (3), $U(x, y)$ is given by

$$U(x, y) = \int A(x, y)\, dx + F(y). \tag{6}$$

The function $F(y)$ can be found from Eq. (4) by differentiating Eq. (6) with respect to y and equating to $B(x, y)$.

Example

Solve

$$x\frac{dy}{dx} + 3x + y = 0.$$

Answer

Rearranging into the form Eq. (2), we have

$$(3x + y)\,dx + x\,dy = 0,$$

i.e. $A(x, y) = 3x + y$ and $B(x, y) = x$. Since $\partial A/\partial y = 1 = \partial B/\partial x$, the equation is exact, and by Eq. (6), the solution is given by

$$\begin{aligned} U(x, y) &= \int (3x + y)\,dx + F(y) = c_1 \\ \Rightarrow \frac{3x^2}{2} + yx + F(y) &= c_1. \end{aligned}$$

Differentiating $U(x, y)$ with respect to y and equating it to $B(x, y) = x$, we obtain $dF/dy = 0$, which integrates to give $F(y) = c_2$. Therefore, letting $c = c_1 - c_2$, the solution to the original ODE is

$$\frac{3x^2}{2} + xy = c.$$

Inexact equations: integrating factors

Equations that may be written in the form

$$A(x,y)\,dx + B(x,y)\,dy = 0 \text{ but for which } \frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x} \tag{7}$$

are known as inexact equations. However the differential $A\,dx + B\,dy$ can always be made exact by multiplying by an integrating factor $\mu(x,y)$ that obeys

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x}. \tag{8}$$

For an integrating factor that is a function of both x and y, there exists no general method for finding it. If, however, an integrating factor exists that is a function of either x or y alone, then Eq. (8) can be solved to find it.

For example, if we assume that the integrating factor is a function of x alone, $\mu = \mu(x)$, then from Eq. (8),

$$\mu \frac{\partial A}{\partial y} = \mu \frac{\partial B}{\partial x} + B \frac{d\mu}{dx}.$$

Rearranging this expression we find

$$\frac{d\mu}{\mu} = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) dx = f(x)\, dx,$$

where we require $f(x)$ also to be a function of x only. The integrating factor is then given by

$$\mu(x) = \exp\left\{\int f(x)dx\right\} \text{where } f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right). \tag{9}$$

Similarly, if $\mu = \mu(y)$, then

$$\mu(y) = \exp\left\{\int g(y)dy\right\} \text{where } g(y) = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right). \tag{10}$$

Eample

Solve

$$\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}.$$

Answer

Rearranging into the form Eq. (7), we have

$$(4x + 3y^2)\,dx + 2xy\,dy = 0, \tag{11}$$

i.e. $A(x, y) = 4x + 3y^2$ and $B(x, y) = 2xy$.
Therefore,

$$\frac{\partial A}{\partial y} = 6y, \quad \frac{\partial B}{\partial x} = 2y,$$

so the ODE is not exact in its present form.
However, we see that

$$\frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = \frac{2}{x},$$

a function of x alone.

Therefore an integrating factor exists that is also a function of x alone and, ignoring the arbitrary constant, is given by

$$\mu(x) = \exp\left\{2\int \frac{dx}{x}\right\} = \exp(2\ln x) = x^2.$$

Multiplying Eq. (11) through by $\mu(x) = x^2$, we obtain

$$(4x^3 + 3x^2y^2)\,dx + 2x^3y\,dy =$$
$$4x^3\,dx + (3x^2y^2\,dx + 2x^3y\,dy) = 0.$$

By inspection, this integrates to give the solution $x^4 + y^2x^3 = c$, where c is a constant.

Linear equations

Linear first-order ODEs are a special case of inexact ODEs and can be written in the conventional form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{12}$$

Such equations can be made exact by multiplying through by an appropriate integrating factor which is always a function of x alone. An integrating factor $\mu(x)$ must be such that

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x), \tag{13}$$

which may then be integrated directly to give

$$\mu(x)y = \int \mu(x)Q(x)\,dx. \tag{14}$$

The required integrating factor $\mu(x)$ is determined by the first equality in Eq. (13),

$$\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + \mu P y,$$

which gives the simple relation

$$\frac{d\mu}{dx} = \mu(x) P(x) \Rightarrow \mu(x) = \exp\left\{ \int P(x)\, dx \right\}. \tag{15}$$

Example

Solve

$$\frac{dy}{dx} + 2xy = 4x.$$

Answer

The integrating factor is given by

$$\mu(x) = \exp\left\{\int 2x\,dx\right\} = \exp x^2.$$

Multiplying through the ODE by $\mu(x) = \exp x^2$, and integrating, we have

$$y \exp x^2 = 4\int x \exp x^2\,dx = 2\exp x^2 + c.$$

The solution to the ODE is therefore given by $y = 2 + c\,\exp(-x^2)$.

Homogeneous equations

Homogeneous equations are ODEs that may be written in the form

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right), \tag{16}$$

where $A(x,y)$ and $B(x,y)$ are homogeneous functions of the same degree. A function $f(x,y)$ is homogeneous of degree n if, for any λ, it obeys

$$f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

For example, if $A = x^2y - xy^2$ and $B = x^3 + y^3$ then we see that A and B are both homogeneous functions of degree 3.

The RHS of a homogeneous ODE can be written as a function of y/x. The equation can then be solved by making the substitution $y = vx$ so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = F(v).$$

This is now a separable equation and can be integrated to give

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}. \tag{17}$$

Eample

Solve

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right).$$

Answer

Substituting $y = vx$, we obtain

$$v + x\frac{dv}{dx} = v + \tan v.$$

Cancelling v on both sides, rearranging and integrating gives

$$\int \cot v \, dv = \int \frac{dx}{x} = \ln x + c_1.$$

But

$$\int \cot v \, dv = \int \frac{\cos v}{\sin v} \, dv = \ln(\sin v) + c_2,$$

so the solution to the ODE is $y = x \sin^{-1} Ax$, where A is a constant.

Isobaric equations

An isobaric ODE is a generalization of the homogeneous ODE and is of the form

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)}, \tag{18}$$

where the RHS is dimensionally consistent if y and dy are each given a weight m relative to x and dx, i.e. if the substitution $y = vx^m$ makes the equation separable.

Example

Solve

$$\frac{dy}{dx} = \frac{-1}{2yx}\left(y^2 + \frac{2}{x}\right).$$

Answer

Rearranging we have

$$\left(y^2 + \frac{2}{x}\right)dx + 2yx\,dy = 0,$$

Giving y and dy the weight m and x and dx the weight 1, the sums of the powers in each term on the LHS are $2m+1$, 0 and $2m+1$ respectively. These are equal if $2m+1=0$, i.e. if $m = -\frac{1}{2}$. Substituting $y = vx^m = vx^{-1/2}$, with the result that $dy = x^{-1/2}dv - \frac{1}{2}vx^{-3/2}dx$, we obtain

$$v\,dv + \frac{dx}{x} = 0,$$

which is separable and integrated to give $\frac{1}{2}v^2 + \ln x = c$. Replacing v by $y\sqrt{x}$, we obtain the solution $\frac{1}{2}y^2x + \ln x = c$.

Bernoulli's equation

Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \text{ where } n \neq 0 \text{ or } 1 \qquad (19)$$

This equation is non-linear but can be made linear by substitution $v = y^{1-n}$, so that

$$\frac{dy}{dx} = \left(\frac{y^n}{1-n}\right)\frac{dv}{dx}.$$

Substituting this into Eq. (19) and dividing through by y^n, we find

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x),$$

which is a linear equation, and may be solved.

Example

Solve

$$\frac{dy}{dx} + \frac{y}{x} = 2x^3y^4.$$

Answer

If we let $v = y^{1-4} = y^{-3}$, then

$$\frac{dy}{dx} = -\frac{y^4}{3}\frac{dv}{dx}.$$

Substituting this into the ODE and rearranging, we obtain

$$\frac{dv}{dx} - \frac{3v}{x} = -6x^3.$$

Multiplying through by the following integrating factor

$$\exp\left\{-3\int \frac{dx}{x}\right\} = \exp(-3\ln x) = \frac{1}{x^3},$$

the solution is then given by

$$\frac{v}{x^3} = -6x + c.$$

Since $v = y^{-3}$, we obtain $y^{-3} = -6x^4 + cx^3$.

Miscellaneous equations

$$\frac{dy}{dx} = F(ax + by + c), \tag{20}$$

where a, b and c are constants, i.e. x and y appear on the RHS in the particular combination $ax + by + c$ and not in any other combination or by themselves. This equation can be solved by making the substitution $v = ax + by + c$, in which case

$$\frac{dv}{dx} = a + b\frac{dy}{dx} = a + bF(v), \tag{21}$$

which is separable and may be integrated directly.

Example

Solve

$$\frac{dy}{dx} = (x + y + 1)^2.$$

Answer

Making the substitution $v = x + y + 1$, from Eq. (21), we obtain

$$\frac{dv}{dx} = v^2 + 1,$$

which is separable and integrates to give

$$\int \frac{dv}{1 + v^2} = \int dx \Longrightarrow \tan^{-1} v = x + c_1.$$

So the solution to the original ODE is $\tan^{-1}(x + y + 1) = x + c_1$, where c_1 is a constant of integration.

Miscellaneous equations (continued)

We now consider

$$\frac{dy}{dx} = \frac{ax + by + c}{ex + fy + g}, \qquad (22)$$

where a, b, c, e, f and g are all constants. This equation my be solved by letting $x = X + \alpha$ and $y = Y + \beta$, where α and β are constants found from

$$a\alpha + b\beta + c = 0 \qquad (23)$$

$$e\alpha + f\beta + g = 0. \qquad (24)$$

Then Eq. (22) can be written as

$$\frac{dY}{dX} = \frac{aX + bY}{eX + fY},$$

which is homogeneous and may be solved.

Example

Solve

$$\frac{dy}{dx} = \frac{2x - 5y + 3}{2x + 4y - 6}.$$

Answer

Let $x = X + \alpha$ and $y = Y + \beta$, where α and β obey the relations

$$\begin{aligned} 2\alpha - 5\beta + 3 &= 0 \\ 2\alpha + 4\beta - 6 &= 0, \end{aligned}$$

which solve to give $\alpha = \beta = 1$. Making these substitutions we find

$$\frac{dY}{dX} = \frac{2X - 5Y}{2X + 4Y},$$

which is a homogeneous ODE and can be solved by substituting $Y = vX$ to obtain

$$\frac{dv}{dX} = \frac{2 - 7v - 4v^2}{X(2 + 4v)}.$$

This equation is separable, and using partial fractions, we find

$$\int \frac{2+4v}{2-7v-4v^2}dv = -\frac{4}{3}\int \frac{dv}{4v-1} - \frac{2}{3}\int \frac{dv}{v+2}$$
$$= \int \frac{dX}{X},$$

which integrates to give

$$\ln X + \frac{1}{3}\ln(4v-1) + \frac{2}{3}\ln(v+2) = c_1,$$

or

$$X^3(4v-1)(v+2)^2 = 3c_1.$$

Since $Y = vX$, $x = X+1$ and $y = Y+1$, the solution to the original ODE is given by $(4y-x-3)(y+2x-3)^2 = c_2$, where $c_2 = 3c_1$.

Higher-degree first-order equations

Higher-degree first-order equations can be written as $F(x, y, dy/dx) = 0$. The most general standard form is

$$p^n + a_{n-1}(x, y)p^{n-2} + \cdots + a_1(x, y)p + a_0(x, y) = 0, \tag{25}$$

where $p = dy/dx$.

Equations soluble for p

Sometime the LHS of Eq. (25) can be factorized into

$$(p - F_1)(p - F_2) \cdots (p - F_n) = 0, \qquad (26)$$

where $F_i = F_i(x, y)$. We are then left with solving the n first-degree equations $p = F_i(x, y)$. Writing the solutions to these first-degree equations as $G_i(x, y) = 0$, the general solution to Eq. (26) is given by the product

$$G_1(x, y) G_2(x, y) \cdots G_n(x, y) = 0. \qquad (27)$$

Example

Solve

$$(x^3 + x^2 + x + 1)p^2 - (3x^2 + 2x + 1)yp + 2xy^2 = 0. \tag{28}$$

Answer

This equation may be factorized to give

$$[(x + 1)p - y][(x^2 + 1)p - 2xy] = 0.$$

Taking each bracket in turn we have

$$\begin{aligned} (x + 1)\frac{dy}{dx} - y &= 0, \\ (x^2 + 1)\frac{dy}{dx} - 2xy = 0, \end{aligned}$$

which have the solutions $y - c(x + 1) = 0$ and $y - c(x^2 + 1) = 0$ respectively. The general solution to Eq. (28) is then given by

$$[y - c(x + 1)][y - c(x^2 + 1)] = 0.$$

Equations soluble for x

Equations that can be solved for x, i.e. such that they may be written in the form

$$x = F(y, p), \tag{29}$$

can be reduced to first-degree equations in p by differentiating both sides with respect to y, so that

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p}\frac{dp}{dy}.$$

This results in an equation of the form $G(y, p) = 0$, which can be used together with Eq. (29) to eliminate p and give the general solution.

Example

Solve

$$6y^2p^2 + 3xp - y = 0. \tag{30}$$

Answer

This equation can be solved for x explicitly to give $3x = y/p - 6y^2p$. Differentiating both sides with respect to y, we find

$$3\frac{dx}{dy} = \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 6y^2\frac{dp}{dy} - 12yp,$$

which factorizes to give

$$(1 + 6yp^2)\left(2p + y\frac{dp}{dy}\right) = 0. \tag{31}$$

Setting the factor containing dp/dy equal to zero gives a first-degree first-order equation in p, which may be solved to give $py^2 = c$. Substituting for p in Eq. (30) then yields the general solution of Eq. (30):

$$y^3 = 3cx + 6c^2. \tag{32}$$

If we now consider the first factor in Eq. (31), we find $6p^2y = -1$ as a possible solution. Substituting for p in Eq. (30) we find the singular solution

$$8y^3 + 3x^2 = 0.$$

Note that the singular solution contains no arbitrary constants and cannot be found from the general solution (32) by any choice of the constant c.

Equations soluble for y

Equations that can be solved for y, i.e. such that they may be written in the form

$$y = F(x, p), \tag{33}$$

can be reduced to first-degree first-order equations in p by differentiating both sides with respect to y, so that

$$\frac{dy}{dx} = p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{dp}{dx}.$$

This results in an equation of the form $G(x, y) = 0$, which can be used together with Eq. (33) to eliminate p and give the general solution.

Example

Solve

$$xp^2 + 2xp - y = 0. \tag{34}$$

Answer

This equation can be solved for y explicitly to give $y = xp^2 + 2xp$. Differentiating both sides with respect to x, we find

$$\frac{dy}{dx} = p = 2xp\frac{dp}{dx} + p^2 + 2x\frac{dp}{dx} + 2p,$$

which after factorizing gives

$$(p+1)\left(p + 2x\frac{dp}{dx}\right) = 0. \tag{35}$$

To obtain the general solution of Eq. (34), we first consider the factor containing dp/dx. This first-degree first-order equation in p has the solution $xp^2 = c$, which we then use to eliminate p from Eq. (34). We therefore find that the general solution to Eq. (34) is

$$(y - c)^2 = 4cx. \tag{36}$$

If we now consider the first factor in Eq. (35), we find this has the simple solution $p = -1$. Substituting this into Eq. (34) then gives

$$x + y = 0,$$

which is a singular solution to Eq. (34).

Clairaut's equation

The Clairaut's equation has the form

$$y = px + F(p), \tag{37}$$

and is therefore a special case of equations soluble for y, Eq. (33).

Differentiating Eq. (37) with respect to x, we find

$$\begin{aligned} \frac{dy}{dx} = p &= p + x\frac{dp}{dx} + \frac{dF}{dp}\frac{dp}{dx} \\ \Rightarrow \frac{dp}{dx}\left(\frac{dF}{dp} + x\right) &= 0. \end{aligned} \tag{38}$$

Considering first the factor containing dp/dx, we find

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = 0 \Rightarrow y = c_1x + c_2. \tag{39}$$

Since $p = dy/dx = c_1$, if we substitute Eq. (39) into Eq. (37), we find $c_1x + c_2 = c_1x + F(c_1)$.

Therefore the constant c_2 is given by $F(c_1)$, and the general solution to Eq. (37)

$$y = c_1 x + F(c_1), \tag{40}$$

i.e. the general solution to Clairaut's equation can be obtained by replacing p in the ODE by the arbitrary constant c_1. Now considering the second factor in Eq. (38), also have

$$\frac{dF}{dp} + x = 0, \tag{41}$$

which has the form $G(x, p) = 0$. This relation may be used to eliminate p from Eq. (37) to give a singular solution.

Example

Solve

$$y = px + p^2. \tag{42}$$

Answer

From Eq. (40), the general solution is $y = cx + c^2$. But from Eq. (41), we also have $2p + x = 0 \Rightarrow p = -x/2$. Substituting this into Eq. (42) we find the singular solution $x^2 + 4y = 0$.