Higher-order ordinary differential equations

A linear ODE of general order n has the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x). \tag{1}$$

If $f(x)=0$ then the equation is called homogeneous; otherwise it is inhomogeneous. The general solution to Eq. (1) will contain n arbitrary constants.

In order to solve any equation of the form (1), we must first find the general solution of the complementary equation:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0. \tag{2}$$

The general solution of Eq. (2) will contain n linearly independent functions, say $y_1(x), y_2(x), \cdots, y_n(x)$. Then the general solution is given by

$$y_c(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x), \tag{3}$$

where the c_m are arbitrary constants that may be determined if n boundary conditions are provided.

For n functions to be linearly independent over an interval, there must not exist any set of constants $c_1, c_2, \ldots, c_n$ such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0 \qquad (4)$$

over that interval except for the trivial case $c_1 = c_2 = \cdots = c_n = 0$.

By repeatedly differentiating Eq. (4) $n-1$ times in all, we obtain n simultaneous equations for $c_1, c_2, \ldots, c_n$:

$$\begin{aligned} c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) &= 0 \\ c_1 y_1'(x) + c_2 y_2'(x) + \cdots + c_n y_n'(x) &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \cdots + c_n y_n^{(n-1)}(x) &= 0, \end{aligned} \qquad (5)$$

where the prime denotes differentiation with respect to x.

The n functions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent over an interval if

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & & & \\ \vdots & & \ddots & \\ y_1^{(n-1)} & & & y_n^{(n-1)} \end{vmatrix} \neq 0 \tag{6}$$

over that interval; $W(y_1, \ldots, y_n)$ is called the Wronskian of the set of functions.

If $f(x) \neq 0$, the general solution of Eq. (1) is given by

$$y(x) = y_c(x) + y_p(x), \tag{7}$$

where $y_p(x)$ is the particular integral, which can be any function that satisfies Eq. (1) directly, provided it is linearly independent of $y_c(x)$.

Linear equations with constant coefficients

If the a_m in Eq. (1) are constants, then we have

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x). \quad (8)$$

Finding the complementary function $y_c(x)$

The complementary function must satisfy

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0 \quad (9)$$

and contain n arbitrary constants. Substituting a solution of the form $y = Ae^{\lambda x}$ into Eq. (9), we arrive at the following auxiliary equation

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0. \quad (10)$$

In general, the auxiliary equation has n roots, say $\lambda_1, \cdots, \lambda_n$. Some roots may be repeated, and some complex. Three main cases are as follows:

1. *All roots real and distinct.* The n solutions to Eq. (9) are $\exp(\lambda_m x)$ for $m = 1$ to n. The complementary function is therefore

$$y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}. \quad (11)$$

2. *Some roots complex.* If one of the roots of the auxiliary equation is complex, say $\alpha + i\beta$, then its complex conjugate is also a root. So, we write

$$\begin{aligned} & c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}(d_1 \cos \beta x + d_2 \sin \beta x) \\ &= A e^{\alpha x} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} (\beta x + \phi) \end{aligned} \quad (12)$$

where A and ϕ are arbitrary constants.

3. *Some roots repeated.* Suppose λ_1 occurs k times $(k > 1)$ as a root of the auxiliary equation. Then, the complementary function is given by

$$\begin{aligned} y_c(x) &= (c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{\lambda_1 x} \\ &\quad + c_{k+1} e^{\lambda_{k+1} x} + \cdots + c_n e^{\lambda_n x}. \end{aligned} \tag{13}$$

If more than one root is repeated, say λ_2 with l occurrence, then the complementary function reads

$$\begin{aligned} y_c(x) &= (c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{\lambda_1 x} \\ &\quad + (c_{k+1} + c_{k+2} x + \cdots + c_{k+l} x^{l-1}) e^{\lambda_2 x} \\ &\quad + c_{k+l+1} e^{\lambda_{k+l+1} x} + \cdots + c_n e^{\lambda_n x}. \end{aligned} \tag{14}$$

Example

Find the complementary function of the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x. \tag{15}$$

Answer

Setting the RHS to zero, substituting $y = Ae^{\lambda x}$ and dividing through by $Ae^{\lambda x}$ we obtain the auxiliary equation

$$\lambda^2 - 2\lambda + 1 = 0.$$

This equation has the repeated root $\lambda = 1$ (twice). Therefore the complementary function is

$$y_c(x) = (c_1 + c_2 x)e^x.$$

Finding the particular integral $y_p(x)$

If $f(x)$ contains only polynomial, exponential, or sine and cosine terms, then by assuming a trial function for $y_p(x)$ of similar form and substituting it into Eq. (9), $y_p(x)$ can be deduced. Standard trial functions are:

1. If $f(x) = ae^{rx}$ then try $y_p(x) = be^{rx}$.
2. If $f(x) = a_1 \sin rx + a_2 \cos rx$ (a_1 or a_2 may be zero) then try $y_p(x) = b_1 \sin rx + b_2 \cos rx$.
3. If $f(x) = a_0 + a_1 x + \cdots + a_N x^N$ (some a_m may be zero) then try $y_p(x) = b_0 + b_1 x + \cdots + b_N x^N$.
4. If $f(x)$ is the sum or product of any of the above then try $y_p(x)$ as the sum or product of the corresponding individual trial functions.

Example

Find a particular integral of the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x.$$

Answer

Assume $y_p(x) = be^x$. However, since the complementary function of this equation is $y_c(x) = (c_1 + c_2 x)e^x$, we see that e^x is already contained in it, as indeed is xe^x. Multiplying the first guess by the lowest necessary integer power of x such that it does not appear in $y_c(x)$, we therefore try $y_p(x) = bx^2 e^x$. Substituting this into the ODE, we find that $b = 1/2$, so the particular integral is given by $y_p(x) = x^2 e^x/2$.

Example

Solve

$$\frac{d^2y}{dx^2} + 4y = x^2 \sin 2x \tag{16}$$

Answer

The auxiliary equation is

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i. \tag{17}$$

Therefore the complementary function is given by

$$y_c(x) = c_1 e^{2ix} + c_2 e^{-2ix} = d_1 \cos 2x + d_2 \sin 2x \tag{18}$$

First, assume that the particular integral is given by

$$(ax^2 + bx + c)(d \sin 2x + e \cos 2x) \tag{19}$$

However, since $\sin 2x$ and $\cos 2x$ already appear in the complementary function, the trial function must be

$$(ax^3 + bx^2 + cx)(d \sin 2x + e \cos 2x). \tag{20}$$

Substituting this into Eq. (14) to fix the constants in Eq. (20), we find the particular integral to be

$$y_p(x) = -\frac{x^3}{12}\cos 2x + \frac{x^2}{16}\sin 2x + \frac{x}{32}\cos 2x. \quad (21)$$

The general solution to Eq. (16) then reads

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= d_1 \cos 2x + d_2 \sin 2x - \frac{x^3}{12}\cos 2x \\ &\quad + \frac{x^2}{16}\sin 2x + \frac{x}{32}\cos 2x. \end{aligned}$$

Linear equations with variable coefficients

The Legendre and Euler linear equations

Legendre's linear equation has the form

$$a_n(\alpha x+\beta)^n \frac{d^n y}{dx^n}+\cdots+a_1(\alpha x+\beta)\frac{dy}{dx}+a_0 y = f(x), \tag{22}$$

where α, β and the a_n are constants, and may be solved by making the substitution $\alpha x + \beta = e^t$. We then have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dt}{dx}\frac{dy}{dt} = \frac{\alpha}{\alpha x+\beta}\frac{dy}{dt} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} = \frac{\alpha^2}{(\alpha x+\beta)^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right), \end{aligned}$$

and so on for higher derivatives.

Therefore we can write the terms of Eq. (22) as

$$
\begin{aligned}
(\alpha x+\beta)\frac{dy}{dx} &= \alpha\frac{dy}{dt}, \\
(\alpha x+\beta)^2\frac{d^2y}{dx^2} &= \alpha^2\frac{d}{dt}\left(\frac{d}{dt}-1\right)y, \\
&\vdots \\
(\alpha x+\beta)^n\frac{d^ny}{dx^n} &= \alpha^n\frac{d}{dt}\left(\frac{d}{dt}-1\right)\cdots \\
&\quad \left(\frac{d}{dt}-n+1\right)y.
\end{aligned}
\tag{23}
$$

Substituting Eq. (23) into the Eq. (22), the latter becomes a linear ODE with constant coefficients,

$$
a_n\alpha^n\frac{d}{dt}\left(\frac{d}{dt}-1\right)\cdots\left(\frac{d}{dt}-n+1\right)y+\cdots
$$
$$
+a_1\alpha\frac{dy}{dt}+a_0y=f\left(\frac{e^t-\beta}{\alpha}\right).
$$

A special case of Legendre's equation, with $\alpha = 1$ and $\beta = 0$, is Euler's equation,

$$a_n x^n \frac{d^n y}{dx^n} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = f(x), \qquad (24)$$

and may be solved by substituting $x = e^t$.

Alternatively, in the special case where $f(x) - 0$ in Eq. (24), substituting $y = x^\lambda$ leads to a simple algebraic equation in λ, which can be solved to yield the solution to Eq. (24).

Example

Solve

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0 \tag{25}$$

by both methods discussed above.

Answer

First, we make the substitution $x = e^t$, which gives

$$\frac{d}{dt}\left(\frac{d}{dt} - 1\right)y + \frac{dy}{dt} - 4y = 0 \Rightarrow \frac{d^2y}{dt^2} - 4y = 0. \tag{26}$$

The general solution of Eq. (25) is therefore

$$y = c_1e^{2t} + c_2e^{-2t} = c_1x^2 + c_2x^{-2}.$$

Since the RHS of Eq. (25) is zero, we can reach the same solution by instead substituting $y = x^\lambda$ into Eq. (25). This gives

$$\lambda(\lambda - 1)x^\lambda + \lambda x^\lambda - 4x^\lambda = 0,$$

which reduces to

$$(\lambda^2 - 4)x^\lambda = 0.$$

This has the solutions $\lambda = \pm 2$, so the general solution is

$$y = c_1 x^2 + c_2 x^{-2}.$$

Exact equations

Sometimes an ODE may be merely the derivative of another ODE of one order lower. If this is the case then the ODE is called exact. The nth-order linear ODE

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \quad (27)$$

is exact if the LHS can be written as a simple derivative,

$$\begin{aligned} & a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y \\ = & \frac{d}{dx}\left[b_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_0(x)y\right]. \end{aligned} \quad (28)$$

For Eq. (28) to hold, we require

$$a_0(x) - a_1'(x) + a_2''(x) + \cdots + (-1)^n a_n^{(n)}(x) = 0, \quad (29)$$

where the prime denotes differentiation with respect to x. If Eq. (29) is satisfied then a straightforward integration leads to a new equation of one order lower.

Example

Solve

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 1. \tag{30}$$

Answer

Comparing with Eq. (27), we have $a_2 = 1 - x^2$, $a_1 = -3x$ and $a_0 = -1$. Therefore, $a_0 - a_1' + a_2'' = 0$, so Eq. (30) is exact and can therefore be written in the form

$$\frac{d}{dx}\left[b_1(x)\frac{dy}{dx} + b_0(x)y\right] = 1. \tag{31}$$

Expanding the LHS of Eq. (31), we find

$$\frac{d}{dx}\left(b_1\frac{dy}{dx} + b_0 y\right) \quad = \quad b_1\frac{d^2y}{dx^2} + (b_1' + b_0)\frac{dy}{dx} + b_0' y. \tag{32}$$

Comparing Eq. (30) and Eq. (32), we find

$$b_1 = 1 - x^2, \quad b_1' + b_0 = -3x, \quad b_0' = -1.$$

These relations integrate consistently to give $b_1 = 1 - x^2$ and $b_0 = -x$, so Eq. (30) can be written as

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx} - xy\right] = 1. \tag{33}$$

Integrating Eq. (33) gives us directly the first-order linear ODE

$$\frac{dy}{dx} - \left(\frac{x}{1-x^2}\right)y = \frac{x+c_1}{1-x^2},$$

which can be solved to give

$$y = \frac{c_1 \sin^{-1} x + c_2}{\sqrt{1-x^2}} - 1.$$

Partially known complementary function

Suppose we wish to solve the nth-order linear ODE

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \quad (34)$$

and we happen to know that $u(x)$ is a solution of Eq. (34) when the RHS is set to zero. By making the substitution $y(x) = u(x)v(x)$, we can transform Eq. (34) into an equation of order $n-1$ in dv/dx.

In particular, if the original equation is of second order, then we obtain a first-order equation in dv/dx. In this way, both the remaining term in the complementary function and the particular integral are found.

Example

Solve

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x. \tag{35}$$

Answer

The complementary function of Eq. (35) is

$$y_c(x) = c_1 \sin x + c_2 \cos x,$$

Let $u(x) = \cos x$ and make the substitution $y(x) = v(x)\cos x$ into Eq. (35). This gives

$$\cos x \frac{d^2v}{dx^2} - 2\sin x \frac{dv}{dx} = \operatorname{cosec} x, \tag{36}$$

Writing Eq. (36) as

$$\frac{d^2v}{dx^2} - 2\tan x \frac{dv}{dx} = \frac{\operatorname{cosec} x}{\cos x}, \tag{37}$$

the integrating factor is given by

$$\exp\left\{-2\int \tan x \, dx\right\} = \exp[2\ln(\cos x)] = \cos^2 x.$$

Multiplying through Eq. (37) by $\cos^2 x$, we obtain

$$\frac{d}{dx}\left(\cos^2 x \frac{dv}{dx}\right) = \cot x,$$

which integrates to give

$$\cos^2 x \frac{dv}{dx} = \ln(\sin x) + c_1.$$

After rearranging and integrating again this becomes

$$\begin{aligned} v &= \int \sec^2 x \ln(\sin x)\, dx + c_1 \int \sec^2 x\, dx \\ &= \tan x \ln(\sin x) - x + c_1 \tan x + c_2. \end{aligned}$$

Therefore the general solution to Eq. (35) is given by

$$y = c_1 \sin x + c_2 \cos + \sin x \ln(\sin x) - x \cos x.$$

variation of parameters

Suppose we wish to find the particular integral of the equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \quad (38)$$

and the complementary function $y_c(x)$ is

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where the functions $y_m(x)$ are known. We assume a particular integral of the form

$$y_p(x) = k_1(x)y_1(x) + k_2(x)y_2(x) + \cdots + k_n(x)y_n(x). \quad (39)$$

Since we have n arbitrary functions $k_1(x), \ldots, k_n(x)$, but only one restriction on them (namely the ODE), we must impose a further $n-1$ constraints. We can choose these constraints to be as convenient as possible.

$$
\begin{aligned}
k_1'(x)y_1(x) + \cdots + k_n'(x)y_n(x) &= 0 \\
k_1'(x)y_1'(x) + \cdots + k_n'(x)y_n'(x) &= 0 \\
&\vdots \\
k_1'(x)y_1^{(n-2)}(x) + \cdots + k_n'(x)y_n^{(n-2)}(x) &= 0 \\
k_1'(x)y_1^{(n-1)}(x) + \cdots + k_n'(x)y_n^{(n-1)}(x) &= \frac{f(x)}{a_n(x)}.
\end{aligned}
\tag{40}
$$

The last of these equations is not a freely chosen constraint, but must be satisfied given the previous $n-1$ constraints and the original ODE.

This choice of constraints is easily justified. Differentiating Eq. (39) with respect to x, we obtain

$$
y_p' = k_1 y_1' + \cdots + k_n y_n' + (k_1' y_1 + \cdots + k_n' y_n).
$$

Let us define the expression in parenthesis to be zero, giving the first equation in Eq. (40).

Differentiating again we find

$$y_p'' = k_1 y_1'' + \cdots + k_n y_n'' + (k_1' y_1' + \cdots + k_n' y_n').$$

Once more, we set the expression in bracket to be zero. We can repeat this procedure. This yields the first $n-1$ equations in Eq. (40). The mth derivative of y_p for $m < n$ is then given by

$$y_p^{(m)} = k_1 y_1^{(m)} + \cdots + k_n y_n^{(m)}.$$

Differentiating y_p once more we find its nth derivative is given by

$$y_p^{(n)} = k_1 y_1^{(n)} + \cdots + k_n y_n^{(n)} + (k_1' y_1^{(n-1)} + \cdots + k_n' y_n^{(n-1)})$$

Substituting the expression for $y_p^{(m)}$, $m = 0$ to n, into the original ODE Eq. (38), we obtain

$$\sum_{m=0}^{n} a_m (k_1 y_1^{(m)} + \cdots + k_n y_n^{(m)})$$
$$+ a_n (k_1' y_1^{(n-1)} + \cdots + k_n' y_n^{(n-1)}) = f(x).$$

Rearranging the sum over m on the LHS, we find

$$\sum_{m=1}^{n} k_m(a_n y_m^{(n)} + \cdots + a_1 y_m' + a_0 y_m)$$
$$+a_n(k_1' y_1^{(n-1)} + \cdots + k_n' y_n^{(n-1)}) = f(x). \qquad (41)$$

But since the functions y_m are solutions of the complementary equation of Eq. (38), we have (for all m)

$$a_n y_m^{(n)} + \cdots + a_1 y_m' + a_0 y_m = 0.$$

Therefore, Eq. (41) becomes

$$a_n(k_1' y_1^{(n-1)} + \cdots + k_n' y_n^{(n-1)}) = f(x),$$

which is the final equation given in Eq. (40).
Eq. (40) can be solved for the functions $k_m'(x)$, which can be integrated to give $k_m(x)$. The general solution to Eq. (38) is then

$$y(x) = y_c(x) + y_p(x) = \sum_{m=1}^{n} [c_m + k_m(x)] y_m(x).$$

Example

Use the variation of parameters to solve

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x, \tag{42}$$

subject to the boundary conditions
$y(0) = y(\pi/2) = 0$.

Answer

The complementary function of Eq. (42) is

$$y_c(x) = c_1 \sin x + c_2 \cos x.$$

We therefore assume a particular integral of the form

$$y_p(x) = k_1(x) \sin x + k_2(x) \cos x,$$

and impose the additional constraints of Eq. (40),

$$\begin{aligned} k_1'(x) \sin x + k_2'(x) \cos x &= 0, \\ k_1'(x) \cos x - k_2'(x) \sin x &= \operatorname{cosec} x. \end{aligned}$$

Solving these equations for $k_1'(x)$ and $k_2'(x)$ gives

$$\begin{aligned} k_1'(x) &= \cos x \ \mathrm{cosec}\, x = \cot x, \\ k_2'(x) &= -\sin x \ \mathrm{cosec}\, x = -1. \end{aligned}$$

Hence, ignoring the constants of integration, $k_1(x)$ and $k_2(x)$ are given by

$$\begin{aligned} k_1(x) &= \ln(\sin x), \\ k_2(x) &= -x. \end{aligned}$$

The general solution to the ODE, Eq. (42), is therefore

$$y(x) = [c_1 + \ln(\sin x)] \sin x + (c_2 - x) \cos x.$$

Applying the boundary conditions $y(0) = y(\pi/2) = 0$, we find $c_1 = c_2 = 0$, so that

$$y(x) = \ln(\sin x) \sin x - x \cos x.$$

Green's functions

Consider the equation,

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \quad (43)$$

and introduce a linear differential operator $\mathcal{L}$ acting on $y(x)$. That is, Eq. (43) is written as

$$\mathcal{L}y(x) = f(x). \quad (44)$$

Suppose that a function $G(x, z)$ exists (The Green's function) such that the general solution to Eq. (44), which obeys some set of imposed boundary conditions in the range $a \leq x \leq b$, is given by

$$y(x) = \int_a^b G(x, z)f(z)\, dz, \quad (45)$$

where z is the integration variable. If we apply the linear differential operator $\mathcal{L}$ to both sides of Eq. (45), we obtain

$$\mathcal{L}y(x) = \int_a^b [\mathcal{L}G(x, z)]f(z)\, dz = f(x). \quad (46)$$

Comparing of Eq. (46) with a standard property of the Dirac delta function,

$$f(x) = \int_a^b \delta(x-z) f(z)\, dz,$$

for $a \leq x \leq b$, shows that for Eq. (46) to hold for any function $f(x)$, we require (for $a \leq x \leq b$)

$$\mathcal{L}G(x,z) = \delta(x-z), \tag{47}$$

i.e. the Green's function must satisfy the original ODE with the RHS set equal to a delta function. $G(x,z)$ may be thought of as the response of a system to a unit impulse at $x = z$.

In addition to Eq. (47), we must impose two further sets of restrictions on $G(x,z)$. The first requires that the general solution $y(x)$ in Eq. (45) obeys the boundary conditions.

The second concerns the continuity or discontinuity of $G(x,z)$ and its derivative at $x=z$, and can be found by integrating Eq. (47) with respect to x over the small interval $[z-\epsilon, z+\epsilon]$ and taking the limits as $\epsilon \to 0$. We then obtain

$$\begin{aligned} \lim_{\epsilon\to 0} \sum_{m=0}^{n} \int_{z-\epsilon}^{z+\epsilon} a_m(x) \frac{d^m G(x,z)}{dx^m} dx \\ = \lim_{\epsilon\to 0} \int_{z-\epsilon}^{z+\epsilon} \delta(x-z) dx \\ = 1. \end{aligned} \tag{48}$$

Since $d^n G/dx^n$ exists at $x=z$ but its value there is infinite, the $(n-1)$th-order derivative must have a finite discontinuity there, whereas all the lower-order derivatives, $d^m G/dx^m$ for $m < n-1$, must be continuous at this point. Therefore the terms containing these derivatives cannot contribute to the value of the integral on the LHS of Eq. (48).

Noting that, apart from an arbitrary constant, $\int (d^m G/dx^m)dx = d^{m-1}G/dx^{m-1}$, we therefore obtain, for $m = 0$ to $n-1$,

$$\lim_{\epsilon \to 0} \int_{z-\epsilon}^{z+\epsilon} a_m(x) \frac{d^m G(x,z)}{dx^m} dx$$

$$= \lim_{\epsilon \to 0} \left[a_m(x) \frac{d^{m-1} G(x,z)}{dx^{m-1}} \right]_{z-\epsilon}^{z+\epsilon} = 0. \qquad (49)$$

Since only the term containing $d^n G/dx^n$ contributes to the integral in Eq. (48), we find on performing the integration that

$$\lim_{\epsilon \to 0} \left[a_n(x) \frac{d^{n-1} G(x,z)}{dx^{n-1}} \right]_{z-\epsilon}^{z+\epsilon} = 1. \qquad (50)$$

Thus, we have the further n constraints that $G(x,z)$ and its derivatives up to order $n-2$ are continuous at $x = z$, but that $d^{n-1}G/dx^{n-1}$ has a discontinuity of $1/a_n(z)$ at $x = z$.

Example

Use Green's functions to solve

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x, \tag{51}$$

subject to the boundary conditions
$y(0) = y(\pi/2) = 0$.

Answer

From Eq. (47), we see that the Green's functions $G(x, z)$ must satisfy

$$\frac{d^2G(x,z)}{dx^2} + G(x,z) = \delta(x-z). \tag{52}$$

The complementary function of Eq. (52) consists of a linear superposition of $\sin x$ and $\cos x$, and must consist of different superpositions on either side of $x = z$ since its $(n-1)$th derivative is required to have a discontinuity there. Therefore we assume the form of the Green's function to be

$$G(x,z) = \begin{cases} A(z)\sin + B(z)\cos x & \text{for } x < z, \\ C(z)\sin x + D(z)\cos x & \text{for } x > z. \end{cases}$$

We now impose the relevant restrictions on $G(x,z)$ in order to determine the functions $A(z),\ldots,D(z)$. The first of these is that $G(x,z)$ should itself obey the homogeneous boundary conditions $G(0,z) = G(\pi/2,z) = 0$. This leads to the conclusion that $B(z) = C(z) = 0$, so we now have

$$G(x,z) = \begin{cases} A(z)\sin x & \text{for } x < z, \\ D(z)\cos x & \text{for } x > z. \end{cases}$$

The second restriction is the continuity conditions given in Eqs. (49), (50). That is, $G(x,z)$ is continuous at $x = z$ and that dG/dx has a discontinuity of $1/a_2(z) = 1$ at this point. Applying these two constraints, we have

$$\begin{aligned} D(z)\cos z - A(z)\sin z &= 0 \\ -D(z)\sin z - A(z)\cos z &= 1. \end{aligned}$$

Solving these equations for $A(z)$ and $D(z)$, we find

$$A(z) = -\cos z, \qquad D(z) = -\sin z.$$

Thus we have

$$G(x,z) = \begin{cases} -\cos z \sin x & \text{for } x < z, \\ -\sin z \cos x & \text{for } x > z. \end{cases}$$

Therefore from Eq. (45), the general solution to Eq. (51) that obeys the boundary conditions $y(0) = y(\pi/2) = 0$ is given by

$$\begin{aligned} y(x) &= \int_0^{\pi/2} G(x,z) \operatorname{cosec} z \, dz \\ &= -\cos x \int_0^x \sin z \operatorname{cosec} z \, dz \\ &\quad -\sin x \int_x^{\pi/2} \cos z \operatorname{cosec} z \, dz \\ &= -x \cos x + \sin x \ln(\sin x), \end{aligned}$$

General ordinary differential equations

Dependent variable absent

If an ODE does not contain the dependent variable y explicitly, but only its derivatives, then the change of variable $p = dy/dx$ leads to an equation of one order lower.

Example

Solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4x \tag{53}$$

Answer

Using the substitution $p = dy/dx$, we have

$$\frac{dp}{dx} + 2p = 4x. \tag{54}$$

The solution to Eq. (54) is therefore

$$p = \frac{dy}{dx} = ae^{-2x} + 2x - 1,$$

where a is a constant. Thus, the general solution to Eq. (53) is

$$y(x) = c_1e^{-2x} + x^2 - x + c_2.$$

Independent variable absent

If an ODE does not contain the independent variable x explicitly, and if we make the substitution $p = dy/dx$, we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{dp}{dx} = \frac{dy}{dx}\frac{dp}{dy} = p\frac{dp}{dy} \\
\frac{d^3y}{dx^3} &= \frac{d}{dx}\left(p\frac{dp}{dy}\right) = \frac{dy}{dx}\frac{d}{dy}\left(p\frac{dp}{dy}\right) \\
&= p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2, \qquad (55)
\end{aligned}$$

and so on for higher-order derivatives. This leads to an equation of one order lower.

Example

Solve

$$1 + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0. \tag{56}$$

Answer

Making the substitution $dy/dx = p$ and $d^2y/dx^2 = p(dp/dy)$, we obtain the first-order ODE

$$1 + yp\frac{dp}{dy} + p^2 = 0,$$

which is separable and the solution is

$$(1 + p^2)y^2 = c_1.$$

Using $p = dy/dx$, we therefore have

$$p = \frac{dy}{dx} = \pm\sqrt{\frac{c_1^2 - y^2}{y^2}},$$

which may be integrated to give the general solution of Eq. (56),

$$(x + c_2)^2 + y^2 = c_1^2.$$

Non-linear exact equations

Example

Solve

$$2y\frac{d^3y}{dx^3} + 6\frac{dy}{dx}\frac{d^2y}{dx^2} = x. \tag{57}$$

We first note that the term $2y\,d^3y/dx^3$ can be obtained by differentiating $2y\,d^2y/dx^2$ since

$$\frac{d}{dx}\left(2y\frac{d^2y}{dx^2}\right) = 2y\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\frac{d^2y}{dx^2}. \tag{58}$$

Rewriting the LHS of Eq. (57) using (58), we are left with $4(dy/dx)(d^2y/dx^2)$, which itself can be written as a derivative

$$4\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[2\left(\frac{dy}{dx}\right)^2\right]. \tag{59}$$

Since we can write the LHS of Eq. (57) as a sum of simple derivatives of other functions, Eq. (57) is exact. Integrating Eq. (57) with respect to x, and using Eq. (58) and (59), gives

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = \int x\,dx = \frac{x^2}{2} + c_1. \qquad (60)$$

Now we can repeat the process to find whether Eq. (60) is itself exact. Considering the term on the LHS of Eq. (60) that contains the highest-order derivative, we find

$$\frac{d}{dx}\left(2y\frac{dy}{dx}\right) = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2.$$

The expression already contain all the terms on the LHS of Eq. (60), so we can integrate Eq. (60) to give

$$2y\frac{dy}{dx} = \frac{x^3}{6} + c_1x + c_2.$$

Hence the solution is

$$y^2 = \frac{x^4}{24} + \frac{c_1x^2}{2} + c_2x + c_3.$$

Isobaric or homogeneous equations

An nth-order isobaric equation is one in which every term can be made dimensionally consistent upon giving y and dy each a weight m, and x and dx each a weight 1. In the special case where the equation is dimensionally consistent with $m = 1$, the equation is called homogeneous. If an equation is isobaric or homogeneous, then the change in dependent variable $y = vx^m$ (or $y = vx$ for the homogeneous case) followed by the change in independent variable $x = e^t$ leads to an equation in which the new independent variable t is absent except in the form d/dt.

Example

Solve

$$x^3 \frac{d^2y}{dx^2} - (x^2 + xy)\frac{dy}{dx} + (y^2 + xy) = 0. \qquad (61)$$

Answer

Assigning y and dy the weight m, and x and dx the weight 1, the weights of the five terms on the LHS of Eq. (61) are, from left to right: $m+1$, $m+1$, $2m$, $2m$, $m+1$. For these weights all to be equal, we require $m = 1$. Since it is homogeneous, we now make the substitution $y = vx$,

$$x\frac{d^2v}{dx^2} + (1-v)\frac{dv}{dx} = 0. \qquad (62)$$

Substituting $x = e^t$ into Eq. (62), we obtain

$$\frac{d^2v}{dt^2} - v\frac{dv}{dt} = 0, \qquad (63)$$

which can be integrated to give

$$\frac{dv}{dt} = \frac{1}{2}v^2 + c_1. \qquad (64)$$

Eq. (64) is separable, and integrates to give

$$\begin{aligned} \frac{1}{2}t + d_2 &= \int \frac{dv}{v^2 + d_1^2} \\ &= \frac{1}{d_1} \tan^{-1} \left(\frac{v}{d_1} \right). \end{aligned}$$

Rearranging and using $x = e^t$ and $y = vx$, we finally obtain the solution to Eq. (61) as

$$y = d_1 x \tan(\frac{1}{2} d_1 \ln x + d_1 d_2).$$

Equations homogeneous in x or y alone

If the weight of x taken alone is the same in every term in the ODE, then the substitution $x = e^t$ leads to an equation in which the new independent variable t is absent. If the weight of y taken alone is the same in every term then the substitution $y = e^v$ leads to an equation in which the new dependent variable v is absent except in the form d/dv.

Example

Solve

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{2}{y^3} = 0.$$

Answer

This equation is homogeneous in x alone, and on substituting $x = e^t$ we obtain

$$\frac{d^2y}{dt^2} + \frac{2}{y^3} = 0,$$

which does not contain the new independent variable t except as d/dt. We integrate this directly to give

$$\frac{dy}{dt} = \sqrt{2(c_1 + 1/y^2)}.$$

This equation is separable, and we find

$$\int \frac{dy}{\sqrt{2(c_1 + 1/y^2)}} = t + c_2.$$

By multiplying the numerator and denominator of the integrand on the LHS by y, we find the solution

$$\frac{\sqrt{c_1 y^2 + 1}}{\sqrt{2} c_1} = t + c_2.$$

Remembering that $t = \ln x$, we finally obtain

$$\frac{\sqrt{c_1 y^2 + 1}}{\sqrt{2} c_1} = \ln x + c_2.$$